The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

- [x] a. A has n real eigenvalues, counting multiplicities.
- b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
- [x] c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
- d. A is orthogonally diagonalizable.

multiplicity means there may be linearly indep. eigenvectors that aren't perpendicular...

(c) Suppose $A \in \mathbb{R}^{n \times n}$ and $A^T = A$.

- Let λ_1 be an eigenvalue of A with eigenvector x_1 $\quad A x_1 = \lambda_1 x_1$
- Let λ_2 be an eigenvalue of A with eigenvector x_2. $\quad A x_2 = \lambda_2 x_2$

where $\lambda_1 \neq \lambda_2$ Then $x_1 \cdot x_2 = 0$

Now,

$$x_1 \cdot A x_2 = A^T x_1 \cdot x_2 = A x_1 \cdot x_2$$

simplify $\qquad$ simplify.

$$x_1 \cdot \lambda_2 x_2 \qquad \lambda_1 x_1 \cdot x_2$$

factor λ_2 out

Therefore $\lambda_2 \, x_1 \cdot x_2 = \lambda_1 x_1 \cdot x_2$ the only way this could be equal when $\lambda_1 \neq \lambda_2$ is for $x_1 \cdot x_2 = 0$

$\swarrow A$ Note $A \in \mathbb{R}^{3\times 3}$ and $A^T = A$.

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Like last page on the exam... Find eigenvalues and the eigenvectors

$$\chi(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$$

switch rows to guarantee the pivot is not zero

$$= -\det \begin{bmatrix} -2 & 6-\lambda & 2 \\ 3-\lambda & -2 & 4 \\ 4 & 2 & 3-\lambda \end{bmatrix}$$

$r_2 \leftarrow r_2 + \frac{3-\lambda}{2} r_1$

$r_3 \leftarrow r_3 + 2 r_1$

$$= -\det \begin{bmatrix} -2 & 6-\lambda & 2 \\ 0 & -2 + \frac{(3-\lambda)(6-\lambda)}{2} & 4 + (3-\lambda) \\ 0 & 2 + 2(6-\lambda) & 3-\lambda+4 \end{bmatrix}$$

$$= -\det \begin{bmatrix} -2 & 6-\lambda & 2 \\ 0 & \frac{\lambda^2 - 9\lambda + 14}{2} & 7-\lambda \\ 0 & 14 - 2\lambda & 7-\lambda \end{bmatrix}$$

$$= \cancel{-}\left(\cancel{-}2 \det \begin{bmatrix} \frac{\lambda^2 - 9\lambda + 14}{2} & 7-\lambda \\ 14 - 2\lambda & 7-\lambda \end{bmatrix}\right)$$

$$= 2(7-\lambda)\left(\frac{\lambda^2-9\lambda+14}{2} - (14-2\lambda)\right)$$

$$= (7-\lambda)\left(\lambda^2-9\lambda+14-28+4\lambda\right)$$

$$= (7-\lambda)\left(\lambda^2-5\lambda-14\right)$$

$$= (7-\lambda)(\lambda-7)(\lambda+2) = 0$$

eigenvalues are $\lambda = 7$ with mult 2

$\lambda = -2$ with mult $\underline{1}$.

expect one free variable

recall: A

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

$\underline{\underline{\lambda = -2}}$ $\quad \text{Nul}(A-\lambda I) = \text{Nul}\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}$

$r_1 \leftrightarrow r_2$

$r_2 \leftrightarrow r_3$

$\begin{bmatrix} -2 & 8 & 2 \\ 4 & 2 & 5 \\ 5 & -2 & 4 \end{bmatrix}$ $\quad r_2 \leftarrow r_2 + 2r_1$, $\quad r_3 \leftarrow r_3 + \frac{5}{2}r_1$ $\quad \begin{bmatrix} -2 & 8 & 2 \\ 0 & 18 & 9 \\ 0 & & \end{bmatrix}$

don't need to compute because I know there is a free vbl

$r_1 \leftarrow \frac{1}{2}r_1$

$r_2 \leftarrow \frac{1}{9}r_2$

$\begin{bmatrix} -1 & 4 & +1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad r_1 \leftarrow r_1 - 2r_2$

$$\begin{array}{ccc} P & P & F \end{array}$$
$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$-x_1 - 1x_3 = 0$

$2x_2 + x_3 = 0$

$x = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} x_3$

eigenvector for $\lambda = -2$

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

$\lambda = 7$ $\quad Nul(A - \lambda I) = Nul \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix}$

since the multiplicity of $\lambda = 7$ is two then there are two free variables. Thus the last two lines must be zero at the end of the elimination...

$$\begin{bmatrix} -4 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad r_1 \leftarrow \frac{1}{2} r_1 \quad \begin{bmatrix} -2 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-2x_1 - x_2 + 2x_3 = 0 \qquad x_1 = \dfrac{-x_2 + 2x_3}{2}$

$$x = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3$$

two eigen vectors... notice they are not perpendicular...

19. $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

eigen values   eigen vectors

-2 $\begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$

dot of these is zero

7 $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

problem

7 $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

dot of these is zero...

Use Gram-Schmidt to fix the problem

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \to Q$$

$$u_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{5}(-1) \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 \\ 2/5 \\ 1 \end{bmatrix}$$

eigen values — eigen vectors

-2 — $\begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$

dot of these is zero

7 — $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

perpendicular

7 — $\begin{bmatrix} 4/5 \\ 2/5 \\ 1 \end{bmatrix}$

still okay