$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$A^T = A$

So the Spectral theorem applies to this matrix

eigen values	eigen vectors
-2	$\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \propto \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$
7	$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$
7	$\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} \propto \begin{bmatrix} 4/5 \\ 2/5 \\ 1 \end{bmatrix}$

do

perpendicular

Make the eigenvectors into unit vectors

$$\lambda_1 = -2, \quad x_1 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \qquad \lambda_2 = 7, \quad x_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \lambda_3 = 7, \quad x_3 = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$\hat{x}_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{9}} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$\hat{x}_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$\hat{x}_3 = \frac{x_3}{\|x_3\|} = \frac{1}{\sqrt{45}} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4/(3\sqrt{5}) \\ 2/(3\sqrt{5}) \\ 5/(3\sqrt{5}) \end{bmatrix}$$

$$\begin{array}{r} 16 \\ 4 \\ \hline 20 \\ 25 \\ 45 \end{array}$$

$$Q = \begin{bmatrix} -2/3 & -1/\sqrt{5} & 4/(3\sqrt{5}) \\ -1/3 & 2/\sqrt{5} & 2/(3\sqrt{5}) \\ 2/3 & 0 & 5/(3\sqrt{5}) \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Since: $AQ = QD$ $\qquad$ $A = QDQ^T$

$$\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 & -1/\sqrt{5} & 4/(3\sqrt{5}) \\ -1/3 & 2/\sqrt{5} & 2/(3\sqrt{5}) \\ 2/3 & 0 & 5/(3\sqrt{5}) \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 4/(3\sqrt{5}) & 2/(3\sqrt{5}) & 5/(3\sqrt{5}) \end{bmatrix}$$

- Singular value decomposition is a way to apply the spectral theorem to an arbitrary matrix

$A \in \mathbb{R}^{m \times n}$ $\qquad$ want to create a related matrix that is square and symmetric so we can apply the spectral...

$$B = \underset{n\times m}{A^T}\ \underset{m\times n}{A} \in \mathbb{R}^{n \times n}$$

Note if A has lin. ind columns then B is invertible... otherwise it's not invertible...
If B is not invertible then $\lambda = 0$ is an eigenvalue of B.

There is nothing very bad about eigenvalues which are zero, but for simplicity the first time through I'll assume B is invertible... ✓

also assume A is square for now...

Note $C = A A^T \in \mathbb{R}^{m \times m}$
$\quad\; {\scriptstyle m\times n \;\; n \times m}$

is another square matrix related to A (also $C = C^T$)

Claim $B = B^T$

$$B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$$

By the Spectral theorem there is an orthonormal basis of eigenvectors for B.

$$Bx_i = \lambda_i x_i \qquad \text{and} \qquad x_i \cdot x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Let

$$U = \begin{bmatrix} x_1 \,|\, x_2 \,|\, \cdots \,|\, x_n \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

note $\Sigma^2 = D$

Then $B = U D U^T$.

What do these x_i's have to do with the original matrix A?

Let $y_i = A x_i$

x_i is an eigenvector of B

$$\text{then } y_i \cdot y_j = Ax_i \cdot Ax_j = A^T A x_i \cdot x_j = \widetilde{B} x_i \cdot x_j$$

$$= \lambda_i x_i \cdot x_j = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Note $\lambda_i = y_i \cdot y_i \geq 0$ $\Rightarrow$ so taking $\sqrt{\lambda_i}$ is no problem...

$$\|y_i\| = \sqrt{y_i \cdot y_i} = \sqrt{\lambda_i}$$

make unit vectors out of the y_i's by defining

$$z_i = \frac{y_i}{\|y_i\|} = \frac{y_i}{\sqrt{\lambda_i}}$$

I assumed none of the eigen-values are zero...

$$y_i = \sqrt{\lambda_i}\, z_i$$

If $\lambda_i = 0$ that means $\|y_i\| = 0$ in which case $y_i = 0$... and you can ignore it... but never mind...

$y_i = Ax_i$, $z_i = \dfrac{Ax_i}{\sqrt{\lambda_i}}$

$$\bar{U} = \left[x_1 \middle| x_2 \middle| \cdots \middle| x_n\right], \quad V = \left[z_1 \middle| z_2 \middle| \cdots \middle| z_n\right]$$

if this V were not square one could just make up some extra vectors and add them to the end...

Now...

$$AU = \left[Ax_1 \middle| Ax_2 \middle| \cdots \middle| Ax_n\right] = \left[y_1 \middle| y_2 \middle| \cdots \middle| y_n\right]$$

$$= \left[\sqrt{\lambda_1} z_1 \middle| \sqrt{\lambda_2} z_2 \middle| \cdots \middle| \sqrt{\lambda_n} z_n\right] = \left[z_1 \middle| z_2 \middle| \cdots \middle| z_n\right] \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \ \sqrt{\lambda_n} \end{bmatrix}$$

$$= V\Sigma \qquad \Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \ \sqrt{\lambda_n} \end{bmatrix}$$

Thus $\quad AU = V\Sigma \quad$ or $\quad A = V\Sigma U^T$

orthogonal matrix

has a lot of geometric meaning $\Rightarrow$ useful in application...

orthogonal matrix. diagonal matrix

7. $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \leftarrow A$

$$B = A^TA = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\det(B - \lambda I) = \det\begin{bmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = (\lambda-8)(\lambda-5) - 4$$

$$= \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9)$$

eigenvalues of $B = A^TA$. $\lambda = 4, 9$

The singular values of A

$$\Sigma = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{9} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$\underline{\underline{\lambda = 4}}$

$$\mathrm{Nul}(B - \lambda I) = \mathrm{Nul}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$$2x_1 + x_2 = 0$$
$$x_1 = -\tfrac{1}{2}x_2$$

$$x = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} x_2$$

eigenvector or $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Need unit eigenvector for V.

$$\hat{x}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$\cancel{\lambda = 9}$

$$\mathrm{Nul}\left(\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} - \lambda I\right) = \mathrm{Nul}\left(\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\right)$$

$$-1x_1 + 2x_2 = 0 \qquad x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2$$

$$x_1 = 2x_2$$

& eigenvector

Unit eigenvector for U

$$\hat{x}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$V = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \quad \text{where} \quad z_i = \frac{Ax_i}{\sqrt{\lambda_i}}$$

finish this next time to find V.