$$A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$$

$x_3 = x_3$

$\lambda = 15$

$$x = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} x_3$$

eigenvector for $\lambda = 15$

$\lambda = -3$

$$A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$$

two free vbls

are zero

dot is zero

$$\mathrm{Nul}(A - \lambda I) = \mathrm{Nul}\begin{bmatrix} 8 & 8 & -4 \\ 8 & 8 & -4 \\ -4 & -4 & 2 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$2x_1 + 2x_2 - x_3 = 0$$

$$x_1 = -x_2 + \tfrac{1}{2}x_3$$

$$x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} x_3$$

Therefore $A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$ has eigenvalues and eigenvectors given by

this dot product is not 0

$$\lambda_1 = 15, \quad x_1 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \qquad \lambda_2 = -3, \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_3 = -3, \quad x_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

dot prod. is 0

dot prod is 0

Gram-schmidt on the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} - \underbrace{\left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right)}_{\text{coefficient that goes in R}} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{2} \left(-\frac{1}{2} \right) \right) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1 \end{bmatrix}$$

rescale it $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

then normalize it

$$v_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Therefore an orthonormal set of eigenvector is

$$\lambda_1 = 15, \quad x_1 = \frac{1}{3} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \qquad \lambda_2 = -3, \quad x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda_3 = -3, \quad x_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

Orthogonal diagonalization of a symmetric matrix

$$D = \begin{bmatrix} 15 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad S = \begin{bmatrix} -2/3 & -1/\sqrt{2} & 1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}$$

Thus $AS = SD$ or $\boxed{A = SDS^{-1}}$ ← note this factorization where S is any invertible matrix happens for most general matrices A but not all of them...

- Since S has orthonormal columns then $S^TS = I$
- Since S is square this means $S^{-1} = S^T$.

Thus $A = SDS^T$ ← orthogonal matrix ← this factorization is the same as before except now S has a geometric meaning... "orthogonal matrix".

diagonal

orthogonal matrix

$$\begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 & -1/\sqrt{2} & 1/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix} \begin{bmatrix} 15 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} -2/3 & -2/3 & 1/3 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & 1/\sqrt{18} & 4/\sqrt{18} \end{bmatrix}$$

The spectral theorem and this factorization is so useful that given a arbitrary matrix A we often try to construct a related matrix that is symmetric...

The singular value decomposition can be viewed as a way to use the spectral theorem for matrices that aren't symmetric...

Given a general matrix $A \in \mathbb{R}^{m \times n}$

$$B = \underset{n\times m}{A^T}\,\underset{m\times n}{A} \in \mathbb{R}^{n \times n} \qquad C = \underset{m\times n}{A}\,\underset{n\times m}{A^T} \in \mathbb{R}^{m \times m}$$

note $C^T = C$ similar to argument for B.

$$B^T = (A^T A)^T = A^T A^{TT} = A^T A = B$$

So B is symmetric...

Now apply spectral theorem to B: Thus B has an orthonormal basis of eigenvectors...

$$B x_i = \lambda_i x_i \qquad x_i \cdot x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for $i = 1, \ldots, n$ ← enough for a basis of $\mathbb{R}^n$

What can I say about A and the vectors x_i?

$$y_i = A x_i$$

← note since x_i's are eigenvectors of B we don't expect them to be eigenvectors of A

$$y_i \cdot y_j = A x_i \cdot A x_j = A^T A x_i \cdot x_j = \underbrace{B x_i}_{x_i \text{ is an eigenvector of } B} \cdot x_j$$

$$= \lambda_i x_i \cdot x_j = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Note

$$y_i \cdot y_i = \lambda_i = \|y_i\|^2 \geq 0$$

Therefore $\|y_i\| = \sqrt{\lambda_i}$

Note: If the columns of A are linearly independent then $A^TA = B$ is invertible

- If B is invertible than none of its eigenvalues are zero...
- Since $\|y_i\| = \sqrt{\lambda_i}$ that means none of the y_i's are zero...

Why: Then $A = QR$ and $A^TA = (QR)^T(QR)$
$= R^TQ^TQR$
$= R^TR$

triangular matrix that's invertible

also invertible

Thus A^TA is invertible

Assume all the $\lambda_i \neq 0$...

$$\boxed{z_i = \frac{y_i}{\|y_i\|} = \frac{y_i}{\sqrt{\lambda_i}}}$$

$i = 1, \dots, n$ is an orthonormal set

$y_i = \sqrt{\lambda_i}\, z_i$

Now

Let $U = \left[x_1 \middle| x_2 \middle| \cdots \middle| x_n \right]$ $\quad V = \left[z_1 \middle| z_2 \middle| \cdots \middle| z_n \right]$

note that $U^TU = I$ and $V^TV = I$

$$AU = A\left[x_1 \middle| x_2 \middle| \cdots \middle| x_n \right] = \left[Ax_1 \middle| Ax_2 \middle| \cdots \middle| Ax_n \right]$$

singular values of A

$$= \left[y_1 \middle| y_2 \middle| \cdots \middle| y_n \right] = \left[\sqrt{\lambda_1} z_1 \middle| \sqrt{\lambda_2} z_2 \middle| \cdots \middle| \sqrt{\lambda_n} z_n \right] = \underbrace{\left[z_1 \middle| z_2 \middle| \cdots \middle| z_n \right]}_{V} \underbrace{\begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \ \sqrt{\lambda_n} \end{bmatrix}}_{\Sigma}$$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \ \sqrt{\lambda_n} \end{bmatrix}$$

Then

$$AU = V\Sigma$$

if U is invertible then $U^{-1} = U^T$

$$\boxed{A = V \Sigma U^T}$$

← called the singular value decomposition

diagonal

orthogonal...

orthogonal

Example...

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

square but not symmetric...

$$B = A^T A = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

eigenvalues of B

$$\det(B - \lambda I) = \det\begin{bmatrix} 8-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = (8-\lambda)(5-\lambda) - 4$$

$$= \lambda^2 - 13\lambda + 40 - 4 = \lambda^2 - 13 + 36 = (\lambda - 9)(\lambda - 4)$$

4, 9

eigenvalues $\lambda = 4, 9$

Singular values of A are the square roots of the eigenvalues of B..

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

$\lambda = 4$ $$\mathrm{Nul}(B - \lambda I) = \mathrm{Nul}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0$$
$$x_1 = -\tfrac{1}{2}x_2$$

$$x = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} x_2$$

↑ eigenvector

unit eigenvector is

$$\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\lambda = 9$ $$\mathrm{Nul}(B - \lambda I) = \mathrm{Nul}\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0$$
$$x_1 = 2x_2$$

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2$$

← eigenvector

unit eigenvector is

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore

$$U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

We know A, U, Σ but need V so that

$$A = V \Sigma U^T$$

could use this to solve for V and it's guaranteed that the V will be orthogonal if you do this...

Alternatively set

$$V = \left[z_1 \middle| z_2 \right] \quad \text{where} \quad z_i = \frac{y_i}{\|y_i\|} = \frac{Ax_i}{\sqrt{\lambda_i}}$$

orthogonal matrix and it's

Find V for next time...