13. Find the SVD of $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$ $\in \mathbb{R}^{2\times 3}$ [*Hint:* Work with A^T.]

$B = \underset{3\times 2}{A^T}\underset{2\times 3}{A} \in \mathbb{R}^{3\times 3}$ or $C = \underset{2\times 3}{A}\underset{3\times 2}{A^T} \in \mathbb{R}^{2\times 2}$

$$C = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \leftarrow U \in \mathbb{R}^{2\times 2} \text{ is eigenvectors of } C$$

$$\begin{array}{r} 17 \\ \times 17 \\ \hline 119 \\ 17 \\ \hline 289 \\ -64 \\ \hline 225 \end{array}$$

by the spectral theorem λ's are real and eigenvectors perpendicular

$$\det(C - \lambda I) = \det\begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} = (17-\lambda)^2 - 64$$

$$= \lambda^2 - 34\lambda + 289 - 64 = \lambda^2 - 34\lambda + 225$$

$$\begin{array}{r} 15 \\ \times 15 \\ \hline 75 \\ 15 \\ \hline 225 \end{array}$$

225 → 15 15 → 3 5 3 5

$$= (\lambda - 9)(\lambda - 25)$$

eigenvalues $\lambda = 9$ and $\lambda = 25$

already know one matrix in the SVD... Σ.

$$A^T = V\Sigma U^T$$

$$\underset{3\times 2}{\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}} = \underset{3\times 3}{V}\underset{3\times 2}{\begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}}\underset{2\times 2}{U^T}$$

$$C = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \qquad \lambda_1 = 25 \quad (\sigma_1 = 5) \qquad \lambda_2 = 9 \quad (\sigma_2 = 3)$$

Find the eigenvectors of C.

Ignore the last row because guaranteed there is one free vbl.

$\lambda = 25$ // $\text{Nul}(C - \lambda I) = \text{Nul}\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$$x_1 - x_2 = 0 \qquad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$
$$x_1 = x_2$$

unit eigenvector for $\lambda = 25$ is $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 9$ $\text{Nul}(C - \lambda I) = \text{Nul}\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$$x_1 + x_2 = 0 \qquad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$$
$$x_1 = -x_2$$

unit eigenvector for $\lambda = 9$ is $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \qquad U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

What is V?

To find V map the eigenvectors in U forward by A^T in this case and then normalize them into unit vectors...

$$\vec{A}x_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} \xrightarrow{\text{normalize to find unit vector}} z_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{A}x_2 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} \xrightarrow{\text{normalize}} z_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & \\ 1/\sqrt{2} & 1/\sqrt{18} & \\ 0 & -4/\sqrt{18} & \end{bmatrix}$$

← what goes here? Any unit vector that is perpendicular to all the other columns...

perpendicular to the first one means

$$\begin{bmatrix} -a \\ a \\ b \end{bmatrix}$$

now perpendicular to the second column to solve for a and b...

$$\begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -a \\ a \\ b \end{bmatrix} = a + a - 4b = 0$$

$$2a - 4b = 0$$

$$a = 2b$$

thus $\frac{1}{\sqrt{9}} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ is a unit vector perpendicular to the others...

$$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{bmatrix}$$

Let put everything together $\quad A^T = V \Sigma U^T$

note this is the eigenvector of $B = A^T A$ with eigenvalue $\lambda_3 = 0$.

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Since we worked with A^T instead of A we ended up with the SVD of A^T...

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

What to do?

Transpose again to find the SVD of A

$$A^T = V \Sigma U^T$$

$$A = A^{TT} = (V \Sigma U^T)^T = U^{TT} \Sigma^T V^T = U \Sigma^T V^T$$

$$\underbrace{\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{\text{orthogonal}} \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}}_{\text{diagonal}} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}}_{\text{orthogonal}}$$

$Ax = b \qquad U \Sigma^T V^T x = b$

Same as these three equation,

$$\begin{cases} Uy = b \\ \Sigma z = y \\ V^T x = z \end{cases}$$

$y = U^T b$

easy to solve diagonal system

$x = Vz$

Can also use to solve least squares... but not this course or today..

$$Ax = b$$ ← x is a solution to $Ax=b$

$$Az = 0$$ ← z solution to homogeneous eq

$$Ax + Az = b + 0$$

$$A(x+z) = b$$

thus $x+z$ is also a solution to $Ax=b$

$\det(I) = 1$ by definition..

Since $I = AA^{-1}$ then $\det(AA^{-1}) = 1$

$$\det AA^{-1} = (\det A)(\det A^{-1}) = 1$$

Cramer's rule $\quad x_i = \dfrac{\det A_i(b)}{\det A}$ solves $Ax=b$

can only divide if $\det A \neq 0$ which means A is invertible