Another example using row operations to find the row-echelon form, solve $Ax=0$ and other things...

There is a zero where the pivot should have been...

$$A = \begin{bmatrix} 0 & 1 & 3 & 2 & -8 \\ 1 & 1 & 5 & -2 & 11 \\ -1 & 2 & 4 & 3 & -10 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$ row swap

$$\begin{bmatrix} 1 & 1 & 5 & -2 & 11 \\ 0 & 1 & 3 & 2 & -8 \\ -1 & 2 & 4 & 3 & -10 \end{bmatrix}$$

$r_3 \leftarrow r_3 + r_1$

undo these elimination steps to get L

pivot

$$\begin{bmatrix} 1 & 1 & 5 & -2 & 11 \\ 0 & \boxed{1} & 3 & 2 & -8 \\ 0 & 3 & 9 & 1 & 1 \end{bmatrix}$$

$r_3 \leftarrow r_3 - 3r_2$

$$\begin{bmatrix} 1 & 1 & 5 & -2 & 11 \\ 0 & 1 & 3 & 2 & -8 \\ 0 & 0 & 0 & -5 & 25 \end{bmatrix} = U$$

Echelon form of A

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

what is the matrix that corresponds to $r_1 \leftrightarrow r_2$?

Compose $r_1 \leftrightarrow r_2$ with the identity matrix to find the matrix for the row swap...

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P function $\rightarrow$ $r_1 \leftrightarrow r_2$

inputs

$$\begin{array}{ccc} x & y & z \\ \end{array}$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = P$$

$PI = P$

know I

know the effect of doing $r_1 \leftrightarrow r_2$ on I

can compute

what is the function $p(x,y,z)$ that corresponds to the matrix P?

$$p\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = p(x,y,z) = (y, x, z) = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

$A = P\,L\,U$ this factorization of A

- U: echelon form (upper triangular)
- L: lower triangular
- P: permutation matrix (permutes the rows)

3 rows and 5 columns

3 rows: output a 3-vector

5 columns: input a 5-vector

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$
$$\begin{bmatrix} 0 & 1 & 3 & 2 & -8 \\ 1 & 1 & 5 & -2 & 11 \\ -1 & 2 & 4 & 3 & -10 \end{bmatrix}$$

Could check if this really worked...

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 5 & -2 & 11 \\ 0 & 1 & 3 & 2 & -8 \\ 0 & 0 & 0 & -5 & 25 \end{bmatrix}}_{U}$$

Use it to solve $Ax = b$ this matrix equation...

means the same as

$$\begin{cases} x_2 + 3x_3 + 2x_4 - 8x_5 = b_1 \\ x_1 + x_2 + 5x_3 - 2x_4 + 11x_5 = b_2 \\ -x_1 + 2x_2 + 4x_3 + 3x_4 - 10x_5 = b_3 \end{cases}$$

3 equations, so the row operations are represented by 3x3 matrices.

Since $A = PLU$ then $Ax = b$ means $P(L(U(x))) = b$

One can solve $Ax = b$ by solving

$$\begin{cases} Py = b \\ Lz = y \\ Ux = z \end{cases}$$

in order

$\checkmark\ y \in \mathbb{R}^3$

The y and z vectors here could be named anything you like and represent intermediate stages of finding the answer...

Section 1.5

The special case when $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$. Then

since P is an invertible function then $Py = 0$ implies $y = 0$.

Since P corresponds to a row operation and all row operations are invertible so the answer to $Py = 0$ is unique and since $P0 = 0$ then it follows that $y = 0$.

since L is an invertible function then $Lz = 0$ implies $z = 0$.

Now solve $Ux = 0$.

$$\underbrace{\begin{bmatrix} 1 & 1 & 5 & -2 & 11 \\ 0 & 1 & 3 & 2 & -8 \\ 0 & 0 & 0 & -5 & 25 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{0}$$

Reduced row echelon form of U and consequently A.

eliminate this

$$\begin{bmatrix} 1 & 1 & 5 & -2 & 11 \\ 0 & 1 & 3 & 2 & -8 \\ 0 & 0 & 0 & -5 & 25 \end{bmatrix} \qquad r_1 \leftarrow r_1 - r_2$$

$$\begin{bmatrix} 1 & 0 & 2 & -4 & 19 \\ 0 & 1 & 3 & 2 & -8 \\ 0 & 0 & 0 & -5 & 25 \end{bmatrix} \qquad r_3 \leftarrow -\frac{1}{5} r_3$$

eliminate these

$$\begin{bmatrix} 1 & 0 & 2 & -4 & 19 \\ 0 & 1 & 3 & 2 & -8 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} \qquad \begin{aligned} r_1 &\leftarrow r_1 + 4r_3 \\ r_2 &\leftarrow r_2 - 2r_3 \end{aligned}$$

$$\begin{array}{ccccc} P & P & F & P & F \end{array}$$
$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} = R$$

reduced row echelon form of A.

Note: $\mathrm{Nul}(A) = \mathrm{Nul}(U) = \mathrm{Nul}(R)$

To solve $Ax=0$ was the same as $Ux=0$ which is now $Rx=0$

$$\begin{cases} x_1 + 2x_3 - 1\cdot x_5 = 0 \\ x_2 + 3x_3 + 2x_5 = 0 \\ x_4 - 5x_5 = 0 \end{cases} \quad \text{thus} \quad \begin{aligned} x_1 &= -2x_3 + x_5 \\ x_2 &= -3x_3 - 2x_5 \\ x_4 &= 5x_5 \end{aligned}$$

Answer is

in vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 + x_5 \\ -3x_3 - 2x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \\ 1 \end{bmatrix} x_5}$$

The set of all vectors of this form where x_3 and x_5 are chosen to be any and all real numbers is called the nullspace of A.

$$\mathrm{Nul}(A) = \{x : Ax = 0\} = \left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \\ 1 \end{bmatrix} x_5 \; ; \; x_3 \in \mathbb{R} \text{ and } x_5 \in \mathbb{R} \right\}$$

Nullspace matrix

$$\mathrm{Nul}(A) = \left\{ \overbrace{\begin{bmatrix} -2 & 1 \\ -3 & -2 \\ 1 & 0 \\ 0 & 5 \\ 0 & 1 \end{bmatrix}} c : c \in \mathbb{R}^2 \right\} \longrightarrow N = \begin{bmatrix} -2 & 1 \\ -3 & -2 \\ 1 & 0 \\ 0 & 5 \\ 0 & 1 \end{bmatrix}$$

Same thing using augmented matrices: Solving $Ax=0$

$$\left[\begin{array}{ccccc|c} 0 & 1 & 3 & 2 & -8 & 0 \\ 1 & 1 & 5 & -2 & 11 & 0 \\ -1 & 2 & 4 & 3 & -10 & 0 \end{array}\right] \xrightarrow{\text{row operations}} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -5 & 0 \end{array}\right]$$

Now rewrite the augmented matrix as a system:

$$\begin{cases} x_1 + 2x_3 - 1\cdot x_5 = 0 \\ x_2 + 3x_3 + 2x_5 = 0 \\ x_4 - 5x_5 = 0 \end{cases} \quad \text{thus} \quad \begin{aligned} x_1 &= -2x_3 + x_5 \\ x_2 &= -3x_3 - 2x_5 \\ x_4 &= 5x_5 \end{aligned}$$