# Chapter 1.7

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \quad (2)$$

- No free variables
- unique soln to $Ax=0$

$Ax=0$

- free variables
- $Ax=0$ has an infinite # of solutions

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$$

$$Ax = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$$

Example: $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ -6 \\ 7 \end{bmatrix}$, $\boxed{p=3}$

$v_1, v_2, v_3 \in \mathbb{R}^4$

$n=4$

columns variables input

rows columns

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 3 & 0 & -6 \\ 4 & 0 & 7 \end{bmatrix} \in \mathbb{R}^{4\times 3}$$

rows correspond to output

Consider $Ax=0$

1. There is exactly one solution since $x=0$ is always a solution then that one solution is $x=0$
2. There are infinitely many solutions.

Suppose $Ax=0$ had at least two solutions.

Call one solution $a$ and the other $b$.

One of those solutions is not equal to zero.

For definiteness suppose $a \neq 0$.

Now consider $c = 2a$. Then $c \neq a$ since $a \neq 0$

consider $d = 3a$ Then $d \neq a$ and $d \neq c$ since $a \neq 0$.

Claim $c, d$ are both solutions.

## Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

The idea with linear dependence is to study the solutions of $Ax=0$, or in other words

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

Suppose $x_p \neq 0$ then I could solve for $V_p$ as

$$V_p = \frac{-1}{x_p}\left(x_1V_1 + x_2V_2 + \cdots + x_{p-1}V_{p-1}\right)$$

Example
$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$Ax = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$$

$$V_1x_1 + V_2x_2 = 0$$

$V_2$ = combination of the other vectors

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \in \mathbb{R}^{n \times p}$$

rows ↓ n, columns ↓ p

There can be at most n pivots, because there are n rows. If there are more columns than pivots there must be free variables.

Why row x columns and not the other way...

$f(x) = Ax \qquad g(x) = Bx$

$(f \circ g)(x) = f(g(x)) = f(Bx) = ABx$

$f : \mathbb{R}^p \to \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^p$

$A \in \mathbb{R}^{n \times p} \qquad B \in \mathbb{R}^{p \times m}$

$$\underset{n \times p}{A}\ \underset{p \times m}{B}\ \underset{m}{x}$$