The Matrix of a Linear Transformation

Matrices $\Longleftrightarrow$ Functions

Write the linear function

3 cols because 3 inputs — 3 rows in matrix because 3 outputs

$$f(x,y,z) = (y,\ 2z,\ 3x-y)$$

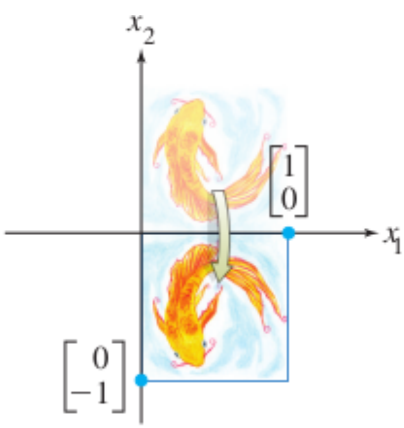
as a matrix

$$\begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & -1 & 0 \end{bmatrix} \end{array}$$

In the book use T instead of f.

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

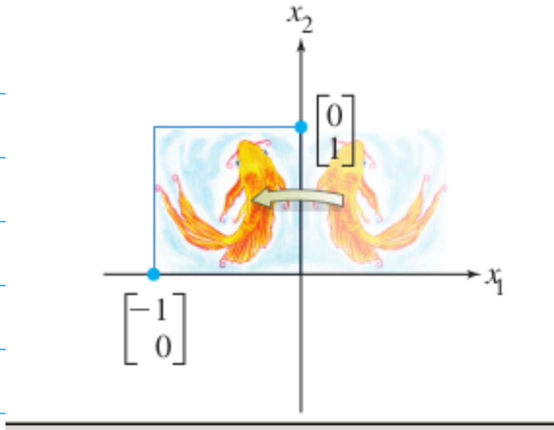
What matrix does this?



$$T(x_1, x_2) = (x_1, -x_2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

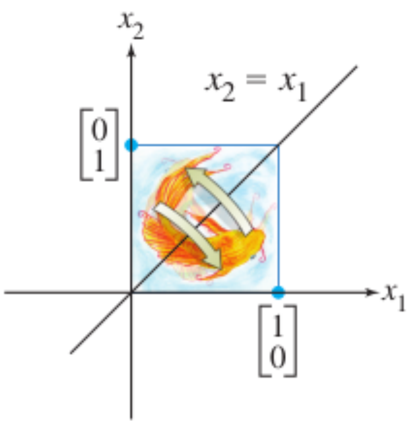
reflection about x_1-axis



$T(x_1, x_2) = (-x_1, x_2)$

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

reflection about the x_2-axis

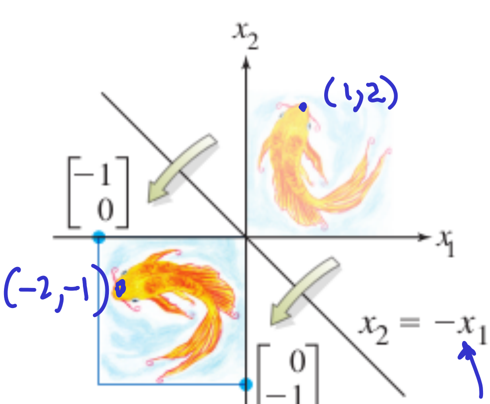


$T(x_1, x_2) = (x_2, x_1)$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

reflection about diagonal

$r_1 \leftrightarrow r_2$



$T(x_1, x_2) = (-x_2, -x_1)$

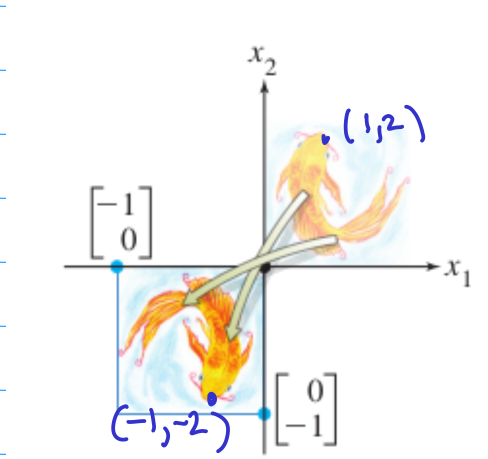
$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

reflection about $x_2 = -x_1$

$\underset{r_2 \leftarrow -r_2}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \underset{r_1 \leftarrow -r_1}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{r_1 \leftrightarrow r_2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

Note any invertible square matrix can be represented by a composition of elementary row operations.

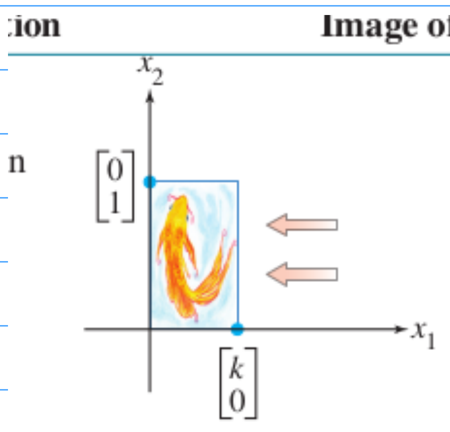
How? Use Gaussian elimination.



reflection through the origin.

$$T(x_1, x_2) = (-x_1, -x_2)$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



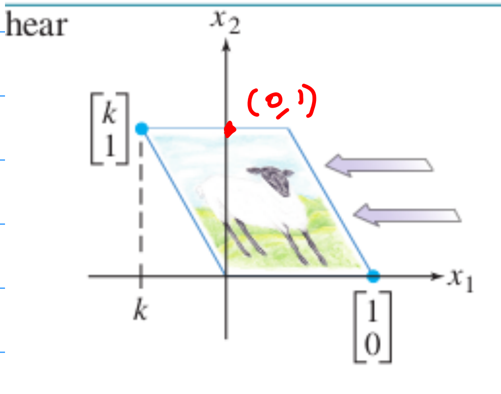
$0 < k < 1$

Horizontal contraction

$$T(x_1, x_2) = (kx_1, x_2)$$

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$r_1 \leftarrow k r_1$



$k<0$

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

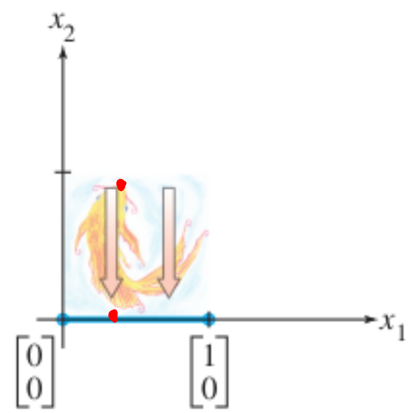
what row operation is this?

$r_1 \leftarrow r_1 + k r_2$

elimination step

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad r_1 \leftarrow r_1 + k r_2 \quad \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Non-invertible



projection onto x_1 axis.

$T(x_1, x_2) = (x_1, 0)$

Projection:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

one pivot
one free variable
$\Rightarrow$ not invertible

can't be written as a composition of elementary row operations

One more linear transform

rotation θ degrees counter clockwise

90^0

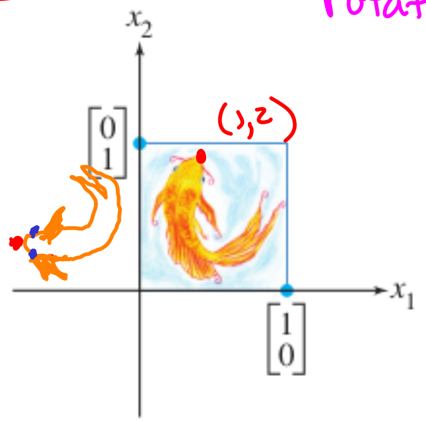


FIGURE 2

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_\theta \Big|_{\theta=90^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

If A then B and If B then A

A

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

B

One-to-one:

If $T(x) = T(y)$ then $x = y$. ← If that holds then $T(x)=0$ has only one solution...

If A then B $\Rightarrow$

Suppose T is one to one and ← A

Suppose $T(x) = 0$ and $T(y) = 0$

Then $T(x) = T(y)$ so $x = y$ so $T(x) = 0$ has only one solution. ← B

If B then A $\Leftarrow$

B start with

Suppose $T(z) = 0$ has only one solution. (the trivial one)

Then since $T(0) = 0$ it follows that $z = 0$.

Now if $T(x) = T(y)$ Then $T(x) - T(y) = 0$

So $T(x-y) = 0$ because T is linear

Let $z = x - y$ so that $T(z) = 0$, Then $z = 0$

or in other words $x - y = 0$ so $x = y$.

then T is one-to-one

conclusion A