Section 1.9

Counter-clockwise rotation

Properties

$T(x)+T(y)=T(x+y)$

$T(\alpha x)=\alpha T(x)$

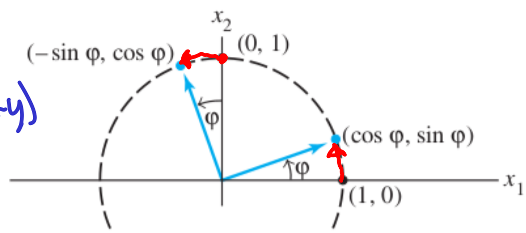


FIGURE 1 A rotation transformation.

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}$$

$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix}$$

linear functions,

A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Let $T(x) = Ax$

- $T\left(\boxed{\begin{bmatrix}1\\0\end{bmatrix}}\right) = \begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}$

$$T\left(\begin{bmatrix}x_1\\0\end{bmatrix}\right) = T\left(x_1\begin{bmatrix}1\\0\end{bmatrix}\right) = x_1 T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = x_1\begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix} = \begin{bmatrix}x_1\cos\varphi\\ x_1\sin\varphi\end{bmatrix}$$

also

- $T\left(\boxed{\begin{bmatrix}0\\1\end{bmatrix}}\right) = \begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix}$

$$T\left(\begin{bmatrix}0\\x_2\end{bmatrix}\right) = T\left(x_2\begin{bmatrix}0\\1\end{bmatrix}\right) = x_2 T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = x_2\begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix} = \begin{bmatrix}-x_2\sin\varphi\\ x_2\cos\varphi\end{bmatrix}$$

Thus

$$T(x) = T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T\left(x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

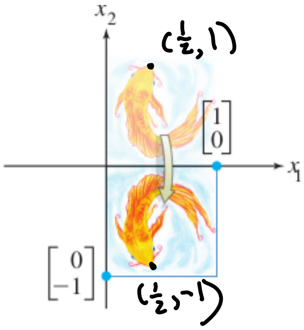
$$= T\left(x_1\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(x_2\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}x_1\cos\varphi\\ x_1\sin\varphi\end{bmatrix} + \begin{bmatrix}-x_2\sin\varphi\\ x_2\cos\varphi\end{bmatrix} = \begin{bmatrix}x_1\cos\varphi - x_2\sin\varphi\\ x_1\sin\varphi + x_2\cos\varphi\end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Notation for those useful vectors:

$$\boxed{e_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad e_2 = \begin{bmatrix}0\\1\end{bmatrix}}$$

reflection about x_1 axis



$$T(x_1, x_2) = (x_1, -x_2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

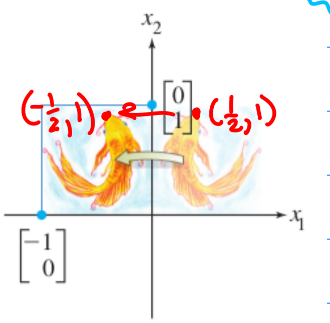
row op... $r_2 \leftarrow -r_2$

How to write a linear function as a matrix: Apply the transformation to the identity matrix and see what you get...

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [e_1 | e_2]$$

- $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T(1,0) = (1,-0) = \begin{bmatrix}1\\0\end{bmatrix}$
- $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T(0,1) = (0,-1) = \begin{bmatrix}0\\-1\end{bmatrix}$

reflection about x_2-axis



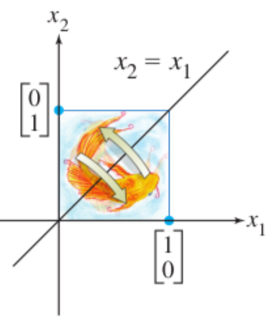
$$T(x_1, x_2) = (-x_1, x_2)$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

row operation ... $r_1 \leftarrow -1 r_1$

reflect about diagonal

elementary row operation



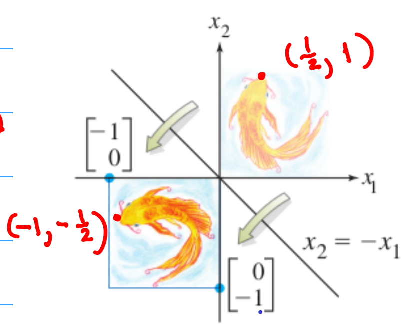
$$T(x_1, x_2) = (x_2, x_1) \approx \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{array}{cc} x_1 & x_2 \\ \end{array} \begin{bmatrix} 0x_1 + 1x_2 \\ 1x_1 + 0x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$

swaps rows

reflect about the other diagonal



$$T(x_1, x_2) = (-x_2, -x_1)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

Invertible transformations

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$r_1 \leftarrow -r_1$
$r_2 \leftarrow -r_2$
$r_1 \leftrightarrow r_1$

can always be written as composition of elementary row operation...

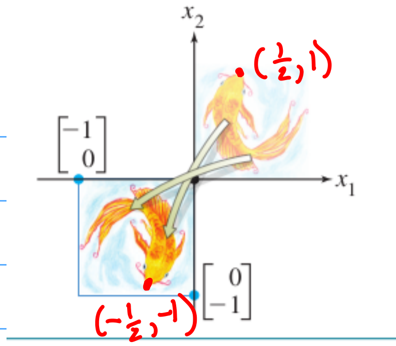
multiply two matrices

$r_1 \leftarrow -r_1$
$r_2 \leftarrow -r_2$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

answer

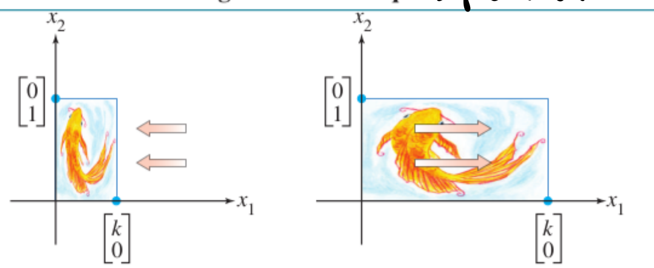
$r_1 \leftrightarrow r_2$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$



$$T(x_1, x_2) = (-x_1, -x_2)$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Horizontal contraction expansion



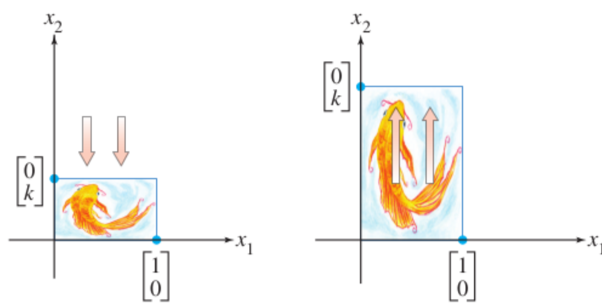
$0 < k < 1$ $k > 1$

$$T(x_1, x_2) = (kx_1, x_2)$$

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$r_1 \leftarrow k r_1$

Vertical contraction and expansion

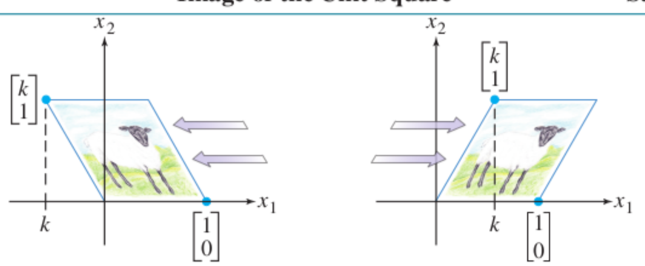


$0 < k < 1$ $k > 1$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

$r_2 \leftarrow k r_2$

Transformation	Image of the Unit Square	St
Horizontal shear		



$k < 0$ $k > 0$

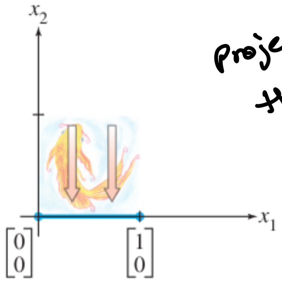
$$T(x_1, x_2) = (x_1 + kx_2, x_2)$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1x_1 + kx_2 \\ 0x_1 + 1x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Elimination step... $r_1 \leftarrow r_1 + k r_2$

Projections



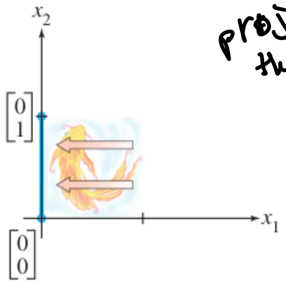
project onto the x_1-axis

$T(x_1, x_2) = (x_1, 0) = \begin{array}{cc} x_1 & x_2 \\ \end{array}$ $\begin{bmatrix} 1x_1 + 0x_2 \\ 0x_1 + 0x_2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

not invertible

not represented in term of row operation because its not invertible



project onto the x_2-axis

$T(x_1, x_2) = (0, x_2) = \begin{bmatrix} 0x_1 + 0x_2 \\ 0x_1 + 1x_2 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$.

can solve. $T(x) = b$ for any b.

solve $Ax = b$ for any b

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$.

If $T(u) = b$ and $T(v) = b$ then $u = v$.

If $Au = Av$ then $u = v$.

$u - v = 0$

$Au = Av \qquad Au - Av = 0 \qquad A(u - v) = 0$

If $Ax = 0$ has a unique solution then A is one-to-one.

The function $T(x)$ corresponding to A is one-to-one is exactly the same as the homogeneous equation $Ax = 0$ having a unique solution.