Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

- One-to-one involves solutions to $Ax=b$ for any b in the range of the function $T(x)=Ax$.
- $T(x)=0$ focuses only on $b=0$.

If equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

then T is one-to-one

and

If T is one-to-one

then equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

☑ First explain why

If equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

then T is one-to-one

Suppose equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

need to show T is one-to-one

Suppose $T(x)=T(y)$ then all I need is to explain why $x=y$ to show T is one-to-one.

Since T is linear can use that.

A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

$T(x) = T(y)$ so $T(x) - T(y) = 0$

substitute

by property (ii) $-T(y) = (-1)T(y) = T(-y)$

Thus $\underbrace{T(x) + T(-y)}_{\text{combine}} = 0$

by property (i) $T(\underbrace{x-y}_{\text{the solution}}) = 0$

Since by hypothesis equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

then $x - y = 0$. Therefore $x = y$ and T is one-to-one

Next explain

If T is one-to-one

then equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Suppose T is one-to-one

need to show equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

need to show $x = 0$.

$T(x) = 0 = T(\underset{y=0}{0})$ Since T is one-to-one by hypothesis $x = 0$

Chapter 2.1

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$(AB)_{ij} = \sum_k a_{ik} b_{kj}$$

move about this later.

Powers of a Matrix

$$A^2 = AA$$

For the output of A to have a dimension suitable to use as the input, A must be square

$$A^3 = AAA$$

$$\vdots$$

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}} = \prod_{i=1}^{k} A$$

With numbers (that is 1x1 matrices)

$$3^2 = 3 \cdot 3.$$

$$3^{1/2} = \sqrt{3}$$

$$3^{p/q} = \sqrt[q]{3^p}$$

$$3^{\sqrt{2}} = e^{\sqrt{2}\ln 3}$$

Calculus

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

$$\boxed{\ln x = \int_1^x \frac{1}{t}\, dt}$$

e^x is inverse of logarithm...

We'll use eigenvectors and eigenvalues to extend A^k in this class...

The Transpose of a Matrix

switch rows with columns.

what's this good for???

Examples

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -2 \end{bmatrix} \in \mathbb{R}^{2\times 3} \qquad A^T = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 3 & -2 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

$$B = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & -2 & 5 \\ -1 & 0 & 2 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 4} \qquad B^T = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 0 \\ 2 & -2 & 2 \\ 3 & 5 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 3}$$

Multiply $\underset{2\times\cancel{3}}{A}\ \underset{\cancel{3}\times 4}{B} = \underset{2\times 4}{C}$

Multiply $\underset{4\times\cancel{3}}{B^T}\ \underset{\cancel{3}\times 2}{A^T} = \underset{4\times 2}{D}$

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & -2 & 5 \\ -1 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & ? & ? & ? \\ ? & ? & -8 & ? \end{bmatrix} = C$$

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 0 \\ 2 & -2 & 2 \\ 3 & 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & ? \\ ? & ? \\ ? & -8 \\ ? & ? \end{bmatrix} = D$$

Conclusion

$C = D^T$

or $C^T = D$

c. For any scalar r, (

$\boxed{\text{d. } (AB)^T = B^T A^T}$

Dot product... (Math 283)

$x = (1, 2, 3)$ $\qquad$ $y = (3, -1, 2)$

$x \cdot y = 1 \cdot 3 + 2 \cdot (-1) + 3 \cdot 2 = 7$

write the vectors as columns in this class

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad y = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$x \cdot y = 7$.

$$x^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

as matrix product

$$x^T y = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 7$$

Inverse of a Matrix 2.2

$Ax = b$ system of linear equations in matrix form

$\boxed{\text{Given } b \text{ find } x}$

If A is invertible then we want to write down its inverse as another matrix.

The inverse of a linear function is a linear function...

Solve $Ax = b$ using augmented matrix

$$\left[\, A \mid b \,\right]$$

What if $b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$b = 2e_1 + 3e_2 + 5e_3$$

Now solve

$$\left[\, A \mid e_1 \,\right] \quad \left[\, A \mid e_2 \,\right] \quad \text{and} \quad \left[\, A \mid e_3 \,\right]$$