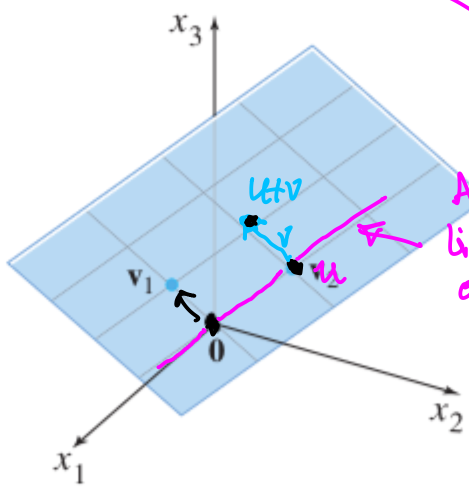
A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

✓ a. The zero vector is in H.

b. For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

✓ c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

eq. of line through u and the origin parameterized by c.



All points on the line through the point v_1 and the origin have to lie in the plane...

FIGURE 1

Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

Column space and Null space of a matrix.

row echelon form of A

$$\textbf{24.}\ A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{array}{cccc} P & F & P & F \\ x_1 & x_2 & x_3 & x_4 \\ \end{array}\begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Solve $Ax=0$ by solving $Ux=0$

$$x_1 - 3x_2 + 6x_3 + 9x_4 = 0$$

$$4x_3 + 5x_4 = 0$$

back substitution

$$x_3 = -\frac{5}{4}x_4$$

same computation

$$x_1 = 3x_2 - 6x_3 - 9x_4 = 3x_2 + \frac{15}{2}x_4 - 9x_4$$

$$x_1 = 3x_2 - \frac{3}{2}x_4$$

vector form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - \frac{3}{2}x_4 \\ x_2 \\ -\frac{5}{4}x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix} x_4$$

free variables

$$\mathrm{Nul}(A) = \{x : Ax = 0\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix} x_4 : x_2, x_4 \in \mathbb{R} \right\}$$

$$\parallel$$

$$\mathrm{Nul}(U) = \{x : Ux = 0\}$$

linear span of two columns...
another way to write this is with a matrix

$$N = \begin{bmatrix} 3 & -3/2 \\ 1 & 0 \\ 0 & -5/4 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{Nul}(A) = \left\{ N \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\} = \{ \underbrace{Nc}_{\text{column space of } N} : c \in \mathbb{R}^2 \}$$

$$= \mathrm{Col}(N)$$

Thus,

$$\mathrm{Nul}\Bigg(\underbrace{\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}}_{A \in \mathbb{R}^{3\times 4}} \Bigg) = \mathrm{Col}\Bigg(\underbrace{\begin{bmatrix} 3 & -3/2 \\ 1 & 0 \\ 0 & -5/4 \\ 0 & 1 \end{bmatrix}}_{N} \Bigg)$$

independent columns...

$$\mathrm{Nul}(A) = \mathrm{Col}(N) \quad \text{or} \quad \{x : Ax = 0\} = \{Nc : c \in \mathbb{R}^2\}$$

What about $\mathrm{Col}(A) = ?$

$\mathrm{Col}\, A = \{ Ax : x \in \mathbb{R}^4 \}$ ← simplify this,,, in order to make a basis by removing the dependent vectors

$$= \left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} x_3 + \begin{bmatrix} -7 \\ 8 \\ 2 \end{bmatrix} x_4 : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

Here is how to simplify

row echelon form of A

$$\textbf{24.}\quad A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{array}{cccc} P & F & P & F \\ x_1 & x_2 & x_3 & x_4 \end{array} \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Keep these and throw away the others

$$\mathrm{Col}\, A = \left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} x_3 : x_1, x_3 \in \mathbb{R} \right\}$$

pivot variables

• still need to define dimension.

$\dim \mathrm{Col}\, A = \#$ of pivot variables $= 2$

$\dim \mathrm{Nul}\, A = \#$ of free variables $= 2$

Note. $\dim \mathrm{Col}\, A + \dim \mathrm{Nul}\, A = 4 = \#$ of columns in A.

Why can you simplify col A like above?

$$\textbf{24.}\quad A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 - \frac{3}{2} r_2$

Find reduced row echelon form...

same computation

$9 - \frac{3}{2}\cdot 5 = \frac{18-15}{2} = \frac{3}{2}$

$$\begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rescale

$r_2 \leftarrow \frac{1}{4} r_2$

reduced row echelon form

$$\begin{array}{cccc} P & F & P & F \end{array}$$
$$\begin{bmatrix} 1 & -3 & 0 & 3/2 \\ 0 & 0 & 1 & 5/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note the free columns are always linear combinations of the pivot columns..

$$\overset{F}{\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}} = -3 \overset{P}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \qquad \overset{F}{\begin{bmatrix} 3/2 \\ 5/4 \\ 0 \end{bmatrix}} = 3/2 \overset{P}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + 5/4 \overset{P}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

These same relationships between the columns exist in the original matrix.

$$\overset{F}{\begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix}} = -3 \overset{P}{\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}} \qquad \overset{F}{\begin{bmatrix} -7 \\ 8 \\ 2 \end{bmatrix}} = 3/2 \overset{P}{\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}} + 5/4 \overset{P}{\begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}}$$

$\frac{3}{2}(-3) + \frac{5}{4}(-2) = \frac{-18}{4} - \frac{10}{4}$

$= \frac{-28}{4} = -7$

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\text{Col } A = \left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} x_3 + \begin{bmatrix} -7 \\ 8 \\ 2 \end{bmatrix} x_4 : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} (x_1 - 3x_2 + \tfrac{3}{2} x_4) + \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} (x_3 + \tfrac{5}{4} x_4) : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} z_1 + \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} z_2 : z_1, z_2 \in \mathbb{R} \right\}$$

the span of the pivot columns..

Are Col A and Nul A really subspaces?

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c. For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

check these properties for each set...

$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \}$ $\qquad$ $\text{Nul } A = \{ x : Ax = 0 \}$

Suppose $u, v \in \text{Col } A$. Need to show that $u + v \in \text{Col } A$.

Since $u \in \text{Col } A$ then $u = Ax$ for some $x \in \mathbb{R}^n$

Since $v \in \text{Col } A$ then $v = Ay$ for some $y \in \mathbb{R}^n$.

Thus $u + v = Ax + Ay = A(x+y)$ by linearity of matrix mult.

This means $u + v \in \text{Col } A$.

Definition

A **basis** for a subspace H of $\mathbb{R}^n$ is a $\boxed{\text{linearly independent}}$ set in H that spans H.

Dimension of a space is the number of vectors in any basis...