$A \in \mathbb{R}^{m \times n}$

### The Rank Theorem

If a matrix $A$ has $n$ columns, then rank $A$ + dim Nul $A = n$.

$n$ variables total

rank $A$ = dim Col$(A)$ = # of pivot variables

dim Nul$(A)$ = # of free variables.

rank $A$ + dim Nul $A$ = $n$

### The Basis Theorem

Let $H$ be a $p$-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly $p$ elements in $H$ is automatically a basis for $H$. Also, any set of $p$ elements of $H$ that spans $H$ is automatically a basis for $H$.

The space $H$ wouldn't even have a dimension without a basis.

Let $\{w_1, w_2, \ldots, w_p\}$ be a basis for $H$.

- The vectors $w_i$ are linearly independent
- The vectors $w_i$ span $H$.

Why?

Suppose $\{v_1, v_2, \ldots, v_p\}$ is a linearly independent set. Claim this set of $v_i$'s is a basis. Need to show they span all of $H$.

Since the $w_i$'s span $H$ and the $v_i \in H$ then each $v_i$ can be written as a combination of the $w_i$'s.

should it be $C_{21}$ or $C_{12}$ ?

$$\begin{cases} v_1 = c_{1,1} w_1 + w_2 c_{2,1} + \cdots + c_{p,1} w_p \\ v_2 = c_{1,2} w_1 + c_{2,2} w_2 + \cdots + c_{p,2} w_p \\ \vdots \\ v_p = c_{1,p} w_1 + c_{2,p} w_2 + \cdots + c_{p,p} w_p \end{cases}$$

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \quad B = \begin{bmatrix} w_1 & w_2 & \cdots & w_p \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pp} \end{bmatrix} \in \mathbb{R}^{p \times p}$$

columns of A are linearly independent...

columns of B are lin-independent ..

$$A = BC$$

has no free variables →

$Ax = 0$ has a unique solution... $x = 0$.

$BCx = 0$ has a unique solution... $x = 0$

Claim C has no free variables...

If C had free variables then $Cx = 0$ would have lots of solutions... And then applying B to both sides would imply $BCx = B0 = 0$ has lots of solutions...

So C has a pivot in every column... Since C is square then it has a pivot in every row as well.

By the invertible Matrix theorem C is invertible...

To show the $v_i$'s span H I need to be able to solve $Ax = b$ for any $b \in H$.

Let $b \in H$. Then since $w_i$'s are a basis, we have

$$b = d_1 w_1 + d_2 w_2 + \cdots + d_p w_p \text{ for some } d_i\text{'s}.$$

Thus $b = Bd$ where

$$B = \begin{bmatrix} w_1 & w_2 & \cdots & w_p \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{bmatrix}$$

Now solve $Ax = Bd$. Note $A = BC$ where $C$ is invertible

$$BCx = Bd$$

That's true, at least, when $Cx = d$. Set $x = C^{-1}d$.

$$Ax = AC^{-1}d = BCC^{-1}d = Bd = b$$

So $Ax = b$ has a solution $x = C^{-1}d$ for every $b \in H$.