Section 1.1 Systems of Linear Equations

linear equation: $y = mx + b$

quadratic equation: $\cancel{ax^2+bx+c=0}$

Transcendental equation: $\cancel{x\cos x = 2}$.

This course is not about these...

Change notation:

$y \leftarrow x_2 \qquad x \leftarrow x_1$

thus

$$x_2 = mx_1 + b$$

$$\boxed{\underbrace{mx_1 - x_2}_{\text{linear function}} = \underbrace{-b}_{\text{right hand side.}}}$$

linear equation

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = mx_1 - x_2$$

Notation

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

tuples of real numbers

2D vectors

set of all two dimensional vectors

$$\boxed{f(x) = -b}$$

What is a linear function?

2 properties

☑ ① $f(x)+f(y)=f(x+y)$ for $x,y\in\mathbb{R}^2$

☑ ② $f(\alpha x)=\alpha f(x)$ for $x\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$

$f(x)=mx_1-x_2$ where $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}\in\mathbb{R}^2$

$f(y)=my_1-y_2$ $y=\begin{bmatrix}y_1\\y_2\end{bmatrix}\in\mathbb{R}^2$

Property ①

$$f(x)+f(y)=(mx_1-x_2)+(my_1-y_2)$$

$$=m(x_1+y_1)-(x_2+y_2)=f\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}\right)=f(x+y)$$

vector arithmetic

$$x+y=\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}$$

section 1.3 in the text...

for $x\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$

$$\alpha x=\alpha\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\alpha x_1\\\alpha x_2\end{bmatrix}$$

The property ②

$$f(\alpha x)=f\left(\begin{bmatrix}\alpha x_1\\\alpha x_2\end{bmatrix}\right)=m(\alpha x_1)-\alpha x_2=\alpha(mx_1-x_2)$$

$$=\alpha f(x)$$

There is one more thing we will do with functions: Composition $f \circ g$. This will correspond to matrix multiplication Chapter 2.

Generalize the linear equation and linear functions

Let $x \in \mathbb{R}^n$ $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

$$f(x) = a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n$$ where the a_i's are coefficients in $\mathbb{R}$.

$f : \mathbb{R}^n \to \mathbb{R}$

$$f(x) = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \cdots + a_{2n} x_n \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + \cdots + a_{3n} x_n \\ \vdots \\ a_{m,1} x_1 + a_{m,2} x_2 + a_{m,3} x_3 + \cdots + a_{m,n} x_n \end{bmatrix}$$

$f : \mathbb{R}^n \to \mathbb{R}^m$

Matrix Notation

$$f(x) = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \\ a_{m1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{x}$$

table of numbers that represents a linear function

simple notation for linear function.

$$f(x) = Ax$$

All functions that satisfy these two properties

✓ ① $f(x) + f(y) = f(x+y)$ for $x, y \in \mathbb{R}^2$

✓ ② $f(\alpha x) = \alpha f(x)$ for $x \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$

can be written as $f(x) = Ax$ for some suitable matrix A.

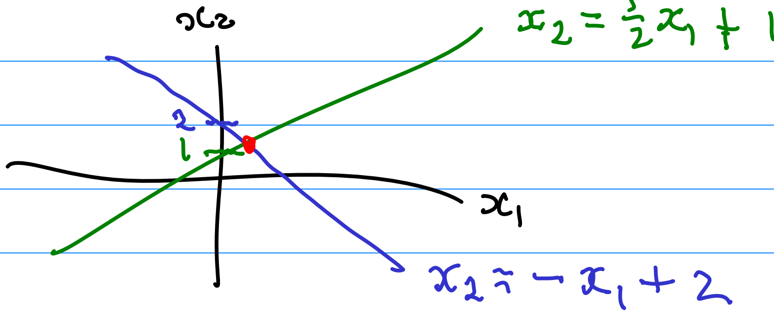
Simple case: System of 2 linear equations

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

Simple notation $Ax = b$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Examples:



simultaneous solution to the system

$$\begin{cases} x_2 = \frac{1}{2}x_1 + 1 \\ x_2 = -x_1 + 2 \end{cases}$$

Write in Matrix notation:

$$\begin{cases} \frac{1}{2}x_1 - x_2 = -1 \\ x_1 + x_2 = 2 \end{cases}$$

$$A = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad Ax = b$$