$$AB = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \cdots & a_1 \cdot b_p \\ \vdots & & & \\ a_m \cdot b_1 & a_m \cdot b_2 & \cdots & a_m \cdot b_p \end{bmatrix} \in \mathbb{R}^{m \times p}$$

$$\underset{2\times 3}{A} = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & -3 \end{bmatrix} \qquad \underset{3\times 2}{B} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\underset{2\times 3}{A}\ \underset{3\times 2}{B} \in \mathbb{R}^{2\times 2}$$

$(2,-1,3)\cdot(-1,1,2) = -2-1+6 = 3$

$(-1,5,-3)\cdot(-1,1,2) = 1+5-6 = 0$

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & -3 \end{bmatrix} \quad \begin{bmatrix} 3 & 3 \\ 0 & 6 \end{bmatrix} = AB$$

Note the matrix that fits here is 2x2. It has the same # of rows as A and the same # of columns as B.

$(2,-1,3)\cdot(1,2,1) = 2-2+3 = 3$

$(-1,5,-3)\cdot(1,2,1) = -1+10-3 = 6$

2.2 The inverse matrix:

Let A be a matrix: $f(x) = Ax$

The linear function corresponding to A:

① Is f invertible?

② What's the inverse?

How to find the inverse (assuming there actually is one):

Let f^{-1} be the inverse of f.

What's the matrix corresponding to f^{-1}?

Note for f^{-1} to be representable by a matrix it must be that f^{-1} is a linear function itself.

Question: Is the inverse of a linear function linear?

☑ ① $f^{-1}(u+v) = f^{-1}(u) + f^{-1}(v)$

② $f^{-1}(\alpha u) = \alpha f^{-1}(u)$

By definition $x = f^{-1}(u)$ means $u = f(x)$

$y = f^{-1}(v)$ means $v = f(y)$

Since f is linear then

$$f(x+y) = f(x) + f(y) = u+v.$$

This is equivalent to saying

$x + y = f^{-1}(u+v)$

Since $x = f^{-1}(u)$ and $y = f^{-1}(v)$ then

$$f^{-1}(u+v) = f^{-1}(u) + f^{-1}(v)$$ so ① is satisfied...

Property ② is similar...

Let f^{-1} be the inverse of f.

What's the matrix corresponding to f^{-1}?

Note for f^{-1} to be representable by a matrix it must be that f^{-1} is a linear function itself. $\boxtimes$

Let's find the matrix. Call A^{-1} the matrix for f^{-1} where $f(x) = Ax$.

$$A^{-1} = \left[f^{-1}(e_1) \middle| f^{-1}(e_2) \middle| \cdots \middle| f^{-1}(e_n) \right]$$

Note, by definition...

$x_1 = f^{-1}(e_1)$ means $e_1 = f(x_1)$

$\vdots$

$x_n = f^{-1}(e_n)$ means $e_n = f(x_n)$

Equivalently solve

$$Ax_1 = e_1, \quad Ax_2 = e_2, \quad \ldots, \quad Ax_n = e_n$$

Need to solve n linear algebra problems to find A^{-1}.

ALGORITHM FOR FINDING A^{-1}

what is I?

Row reduce the augmented matrix $[A \quad I]$. If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A does not have an inverse.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \quad \text{find } A^{-1}.$$

Recall $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

To solve $Ax_1 = e_1$ use $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ -3 & 1 & 4 & 0 \\ 2 & -3 & 4 & 0 \end{array}\right]$.

$Ax_2 = e_2$ use $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ -3 & 1 & 4 & 1 \\ 2 & -3 & 4 & 0 \end{array}\right]$.

$Ax_3 = e_3$ use $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ -3 & 1 & 4 & 0 \\ 2 & -3 & 4 & 1 \end{array}\right]$

Note the vector "b" on the right side doesn't affect the pivots or row operations needed to rewrite the left side in reduced echelon form...

Thus, we can make a "larger augmented matrix

$$[A|e_1|e_2|e_3] = [A|I]$$

where $I = [e_1|e_2|e_3]$,

Remark: we'll see this matrix again. It's called the identity. But for now its just all the right hand sides bunched together...

order is the order I'll need later for A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} r_2 \leftarrow r_2 + 3r_1 \\ r_3 \leftarrow r_3 - 2r_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \quad r_3 \leftarrow r_3 + 3r_2$$

Echelon form

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \quad \begin{array}{l} r_1 \leftarrow r_1 + r_3 \\ r_2 \leftarrow r_2 + r_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \quad r_3 \leftarrow \frac{1}{2} r_3$$

reduced echelon form

$$I \rightarrow \left[\begin{array}{ccc|ccc} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{matrix} \end{array}\right]$$

Read off solutions to

$Ax_1 = e_1$, $Ax_2 = e_2$, $Ax_3 = e_3$

$$x_1 = \begin{bmatrix} 8 \\ 10 \\ 7/2 \end{bmatrix} \quad x_2 = \begin{bmatrix} 3 \\ 4 \\ 3/2 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$$

$$A^{-1} = \left[f^{-1}(e_1) \middle| f^{-1}(e_2) \middle| f^{-1}(e_3) \right] = \left[x_1 \middle| x_2 \middle| x_3 \right] = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$$

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[A \quad I]$. If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A does not have an inverse.

Special case of 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Augmented matrix

$$[A|I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$r_2 \leftarrow r_2 - \frac{c}{a} r$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{c}{a} b & -\frac{c}{a} & 1 \end{array}\right]$$

simplify $\frac{ad-cb}{a}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-cb}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

rescale

$r_2 \leftarrow \frac{a}{ad-cb} r_2$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{array}\right]$$

after finishing this we get

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

Theory of determinants generalizing this...