Matrix factorization $A = LU$

Recall we've been solving $Ax = b$

U is the echelon form from before..

How? We did row operations to find the Echelon form and more row operations to find the reduced echelon form...

L comes from the row operations needed to find U with one limitation. Only use elimination steps in the row operations.

Not being able to use rescaling operations is not a problem. because they are optional when making the echelon form. (rescaling was more for the reduced echelon form). But the row swaps are sometimes necessary to obtain the echelon form. That is when there is a 0 in the pivot position.

Note not all matrices can be factored $A = LU$.

The algorithm is the same, but the interpretation is factoring $A = LU$ rather than solving $Ax = b$.

Example

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$$

① $r_2 \leftarrow r_2 + r_1$

② $r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} \boxed{3} & -7 & -2 \\ 0 & \boxed{-2} & -1 \\ 0 & \cancel{10} & 4 \end{bmatrix}$$

③ $r_3 \leftarrow r_3 + 5r_2$

Echelon form

$$U = \begin{bmatrix} \boxed{3} & -7 & -2 \\ 0 & \boxed{-2} & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

What is L? It's the stuff that has to be done to U to get back to A.

① $r_2 \leftarrow r_2 + r_1$

② $r_3 \leftarrow r_3 - 2r_1$

③ $r_3 \leftarrow r_3 + 5r_2$

Undo the row operations in reverse order to get back to A.

$$A = \underbrace{\begin{bmatrix} r_2 \leftarrow r_2 - r_1 \end{bmatrix}\begin{bmatrix} r_3 \leftarrow r_3 + 2r_1 \end{bmatrix}\begin{bmatrix} r_3 \leftarrow r_3 - 5r_2 \end{bmatrix}}_{\text{this is } L}\begin{bmatrix} \boxed{3} & -7 & -2 \\ 0 & \boxed{-2} & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} r_3 \leftarrow r_3 - 5r_2 \end{bmatrix} = \begin{bmatrix} r_3 \leftarrow r_3 - 5r_2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} r_3 \leftarrow r_3 + 2r_1 \end{bmatrix} = \begin{bmatrix} r_3 \leftarrow r_3 + 2r_1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$[r_2 \leftarrow r_2 - r_1] = [r_2 \leftarrow r_2 - r_1]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}$$

Substitute

$$A = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}\begin{bmatrix}\boxed{3}&-7&-2\\0&\boxed{-2}&-1\\0&0&-1\end{bmatrix}$$

$$L = \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\left(\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}\right)$$

$$= \begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&-5&1\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\2&-5&1\end{bmatrix}$$

Note that the magic where these coefficients in the inverse elimination steps simply plop here depends on the ordering of the elimination steps in the original algorithm from left to right.

Therefore: $A = LU$.

$$\begin{bmatrix}\boxed{3}&-7&-2\\-3&5&1\\6&-4&0\end{bmatrix} = \begin{bmatrix}1&0&0\\-1&1&0\\2&-5&1\end{bmatrix}\begin{bmatrix}\boxed{3}&-7&-2\\0&\boxed{-2}&-1\\0&0&-1\end{bmatrix}$$

Why is $A=LU$ useful? Because L and U are simpler than A because of the 0's in them.

Consider $Ax=b$. We know $A=LU$

original problem — a complicated system of linear equations.

Thus $\underbrace{LUx}_{y}=b$

$$\begin{cases} Ux=y \\ Ly=b \end{cases}$$

two simpler systems of linear equations.

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

first solve $Ly=b$ for y. Plug y in here

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Then solve $Ux=y$ for x and that's the answer...

To solve these systems no more elimination steps are needed...

Suppose solving $Ax=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solve by substitution

$y_1 = 1$

$-y_1 + y_2 = 0$

$2y_1 - 5y_2 + y_3 = 0$

or

$y_1 = 1$

$y_2 = y_1 = 1$

$y_3 = -2y_1 + 5y_2 = -2 + 5 = 3$

$$y = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Next solve $Ux = y$

work backwards

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

back substitution

$-x_3 = 3$

$-2x_2 - x_3 = 1$

$3x_1 - 7x_2 - 2x_3 = 1$

$x_3 = -3$

$x_2 = \dfrac{1 + x_3}{-2} = \dfrac{1-3}{-2} = 1$

$x_1 = \dfrac{1 + 7x_2 + 2x_3}{3} = \dfrac{1 + 7 - 6}{3} = \dfrac{2}{3}$

answer to $Ax = \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}$ is $x = \begin{bmatrix} 2/3 \\ 1 \\ -3 \end{bmatrix}$.

We'll skip 2.6 and 2.7 applications. Please look at those sections to see what they are about.

We'll start with 2.8 next week.