To define the determinant of a large matrix inductively in terms of smaller matrices, we need a way to create smaller matrices from larger ones.

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

A_{ij} is the submatrix of A formed by crossing out the ith row and jth column.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Thus

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Theorem: You don't need to cross out the first row, you could cross out a different one.

$$\det A = \sum_{j=1}^{n} (-1)^{2+j} a_{2j} \det A_{2j}$$

gives the same answer, In general, let i be a row of A then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Theorem: Could expand along a column just as well. Thus,

$$\det A = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1}$$

gives the same answer.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

switch rows and columns

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - cb = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Generalize to let j be a row of A, then

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

gives the same result.

In summary

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$ holding i fixed

or

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$ holding j fixed

all give the same answer.

Note that $\det A^T = \det A$. Because one can switch the roles of rows and columns ...

What is the determinant A a triangular matrix?

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$ What is $\det A$?

$$\det A = \sum_{j=1}^{4} (-1)^{1+j} a_{1j} \det A_{1j}$$

$$= (-1)^2 a_{11} \det A_{11} + \underbrace{(-1)^3 a_{12} \det A_{12} + (-1)^4 a_{13} \det A_{13} + (-1)^5 a_{14} \det A_{14}}_{\text{these are all zero so I won't worry about them.}}$$

$$A_{11} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 5 & 4 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A_{14} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 5 & 4 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

Thus

$$\det \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} = (-1)^2 a_{11} \det A_{11} = 1 \cdot 2 \cdot \det \begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

what's this?

$$\det \begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix} = (-1)^2 \cdot 3 \cdot \det \begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$+ (-1)^3 \cdot 0 \det \begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$+ (-1)^4 \cdot 0 \det \begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

These are zero

$$\det\begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix} = (-1)^2 \cdot 3 \cdot \det\begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}$$

recall

$$\det\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} = (-1)^2 a_{11} \det A_{11} = 1 \cdot 2 \cdot \det\begin{bmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= 2 \cdot 3 \cdot \det\begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix} = 2 \cdot 3 \cdot (-1) \cdot 2 .$$

$$\det\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} = 2 \cdot 3 \cdot (-1) \cdot 2 ,$$

Taking the determinant of a triangular matrix is easy because of all the zeros.

What about

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = 1 \cdot 0 - 2 \cdot 3 = -2 \cdot 3$$

not upper triangular nor lower triangular ...but has some triangle nature...

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \qquad r_1 \leftrightarrow r_2$$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \leftarrow \text{lower triangular} \ldots$$

$$\det \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} = 3 \cdot 2 \quad \text{and} \quad \det \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = -2 \cdot 3$$

Observation: Swapping rows changed the sign of the determinant.

It's true in general...

$A \in \mathbb{R}^{n \times n}$

$$\boxed{\det\left(\left[\underset{i \neq j}{r_i \leftrightarrow r_j}\right] A\right) = -\det A}$$

3.2 Properties of Determinants

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

(a) Elimination step $r_i \leftarrow r_i - \alpha r_j$ $\quad \alpha \neq 0$, $i \neq j$

$$\det\left(\left[r_i \leftarrow r_i - \alpha r_j\right] A\right) = \det A$$

(b) row swap $r_i \leftrightarrow r_j$ $\quad i \neq j$

$$\det\left(\left[r_i \leftrightarrow r_j\right] A\right) = -\det A$$

(c) rescaling $r_i \leftarrow k r_i$ where $k \neq 0$

$$\det\left(\left[r_i \leftarrow k r_i\right] A\right) = k \det A$$