Definition/Theorem on determinants.

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{holding } i \text{ fixed}$$

or

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{holding } j \text{ fixed}$$

i-j cofactor $\quad C_{ij} = (-1)^{i+j} \det A_{ij}$

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad \text{for } i \text{ fixed}$$

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad \text{for } j \text{ fixed}$$

Determinant of a upper or lower triangular matrix is easy

$$\det \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 5 & 4 & -1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} = 2 \cdot 3 \cdot (-1) \cdot 2$$

(a) Elimination step $r_i \leftarrow r_i - \alpha r_j$ $\alpha \neq 0$, $i \neq j$

$$\det\left(\left[r_i \leftarrow r_i - \alpha r_j\right] A\right) = \det A$$

(b) row swap $r_i \leftrightarrow r_j$ $i \neq j$

$$\det\left(\left[r_i \leftrightarrow r_j\right] A\right) = -\det A$$

(c) rescaling $r_i \leftarrow k r_i$ where $k \neq 0$

$$\det\left(\left[r_i \leftarrow k r_i\right] A\right) = k \det A$$

To find the determinant of any matrix use row operations to make the echelon form and then multiply along the diagonal...

Elimination steps don't change the determinant. The others do in predicable ways. The one has to work back to find the $\det A$ from $\det U$ where U is the echelon form of A.

Examples:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}$$

$r_3 \leftarrow r_3 + 2r_1$

$$U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

$U = \left[r_3 \leftarrow r_3 + 2r_1\right] A$

$\det \left[r_3 \leftarrow r_3 + 2r_1\right] A = \det U$.

$\det U = 1 \cdot 1 \cdot 5$

Thus $\boxed{\det A = 5}$

Saves work to do it this way...

It saves a lot of work. In general the recursive definition computes a $n \times n$ determinant from n determinants of size $n-1 \times n-1$, and so forth...

In the end one has $n!$ number of terms.

Gaussian elimination

pivot	1	2	3	. . . .	n
elimination steps	$n-1$	$n-2$	$n-3$	. . . . . .	0

Total elimination steps $= 1 + 2 + \dots + n-1 = \dfrac{(n-1)n}{2} \approx \dfrac{1}{2}n^2$

Each row operation has n terms in it,

Total # of terms $\frac{1}{2}n^3$.

The point is that $\frac{1}{2}n^3$ is much smaller that $n!$ when n is large

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julia> n=10
10

julia> 1/2*n^3
500.0   <- big

julia> factorial(n)
3628800   <- much much bigger ...
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Example

$$A = \begin{bmatrix} 5 & 7 & -2 \\ 1 & 2 & -1 \\ 0 & -6 & 6 \end{bmatrix}$$

changes the sign. $\rightarrow$ $r_1 \leftrightarrow r_2$

These don't change the determinant $\left\{ \begin{array}{l} r_2 \leftarrow r_2 - 5r_1 \\ r_3 \leftarrow r_3 - 2r_2 \end{array} \right.$

Echelon form

$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\det [r_1 \leftrightarrow r_2] A = -\det A$

$\det U = 1 \cdot (-3) \cdot 0 = 0$

Thus $\det A = -\det U = -0 = 0$.

Remark $\det A = 0$ exactly when there is a row (or more) of zeros in the echelon form.

In this case you are missing pivots (i.e. there is not one in every row) so the matrix A is not invertible.

Theorem:

A square matrix A is invertible if and only if $\det A \neq 0$.

Example:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$r_2 \leftarrow r_2 + 3r_1$
$r_3 \leftarrow r_3 - 2r_1$

$r_3 \leftarrow r_3 + 3r_2$

Echelon form

$$U = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

all elimination steps so they didn't affect the determinant.

$\det A = \det U = 2$

Rule for taking determinant of a 3x3 matrix

$$\begin{array}{ccc} -12 & 0 & -4 \\ (1)(-3)(4) & (-3)(0)4 & 2(1)(-2) \end{array} \Bigg\}\ \text{subtract these}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$(-18+4)-(-12-4)$$

$$=-18+20=2$$

same answer as before.

$$\underbrace{0\cdot 4\cdot 2}_{0} \qquad \underbrace{(-3)(-3)(-2)}_{-18} \qquad \underset{4}{1\cdot 1\cdot 4} \quad \Bigg\}\ \text{add these together}$$

Note this is a 3x3 matrix and there are $3! = 6$ terms in the above calculation

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Cramer's rule:

{Next time}

Assume that I already know that
$\det(AB) = (\det A)(\det B)$

Row operations

$$\det(\underbrace{[r_i \leftarrow r_i - \alpha r_j]\, A}_{\text{prod of matrixe}}) = \det A = \underbrace{\det([r_i \leftarrow r_i - \alpha r_j])\det A}_{\text{det of a product}}$$

$$\det\,[r_3 \leftarrow r_3 - 2r_1] = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = 1$$