Solve by Cramer's rule.

3. $3x_1 - 2x_2 = 3$
$-4x_1 + 6x_2 = -5$

$$A = \begin{bmatrix} 3 & -2 \\ -4 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$x_i = \frac{\det A_i(b)}{\det A}$$

$$\det A_1(b) = \det \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix} = 3 \cdot 6 - (-2)(-5) = 18 - 10 = 8$$

$$\det A_2(b) = \det \begin{bmatrix} 3 & 3 \\ -4 & -5 \end{bmatrix} = 3(-5) - (3)(-4) = -15 + 12 = -3$$

$$\det A = \det \begin{bmatrix} 3 & -2 \\ -4 & 6 \end{bmatrix} = 3 \cdot 6 - (-2)(-4) = 18 - 8 = 10$$

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{8}{10} = \frac{4}{5} \qquad x_2 = \frac{\det A_2(b)}{\det A} = \frac{-3}{10}$$

Answer $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -3/10 \end{bmatrix}$

Solve using augmented matrix (for comparison)

$$[A|b] = \begin{bmatrix} 3 & -2 & 3 \\ -4 & 6 & -5 \end{bmatrix}$$

$r_2 \leftarrow r_2 + \frac{4}{3} r_1$

$6 - \frac{8}{3} = \frac{18-8}{3} = \frac{10}{3}$

$-5 + 4 = -1$

$$\begin{bmatrix} 3 & -2 & 3 \\ 0 & 10/3 & -1 \end{bmatrix}$$

$r_2 \leftarrow \frac{3}{10} r_2$

$$\begin{bmatrix} 3 & -2 & 3 \\ 0 & 1 & -\frac{3}{10} \end{bmatrix}$$

$r_1 \leftarrow r_1 + 2r_2$

$$3 - \frac{6}{10} = 3 - \frac{3}{5} = \frac{15-3}{5} = \frac{12}{5}$$

$$\begin{bmatrix} 3 & 0 & 12/5 \\ 0 & 1 & -\frac{3}{10} \end{bmatrix}$$

$r_1 \leftarrow \frac{1}{3} r_1$

$$\begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & -\frac{3}{10} \end{bmatrix} \qquad x = \begin{bmatrix} 4/5 \\ -3/10 \end{bmatrix}$$

When the size of the matrix is larger, then Cramer's rule is proportionately more and more work compared to Gauss-elimination

What can we do with Cramer's rule? Get a closed form solution for the inverse. (Theoretical result because not practical for large matrices).

For example the matrix may have variables & other parameters in it and you want to see how those parameters affect the solution. Then use Cramer's rule.

$$f(x) = Ax \qquad f^{-1}(x) = A^{-1}x$$

$$A = \begin{bmatrix} f(e_1) \,\Big|\, f(e_2) \,\Big|\, \cdots \,\Big|\, f(e_n) \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} f^{-1}(e_1) \,\Big|\, f^{-1}(e_2) \,\Big|\, \cdots \,\Big|\, f^{-1}(e_n) \end{bmatrix}$$

Compute these by solving $Ax = b$ for $b = e_1, e_2, e_3, \ldots, e_n$

Idea: use Cramer's rule to compute $f^{-1}(e_j)$ for $j = 1, \ldots, n$

$A \in \mathbb{R}^{n \times n}$

column of A

$Ax = b$ by Cramer's rule $x_i = \dfrac{\det A_i(b)}{\det A}$

ith entry of the column vector is like the row

Note i is about rows on the left
i is about columns on the right
that's ok because A is square, but weird..

Given A find A^{-1}:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & -7 \\ 3 & 2 & 1 \end{bmatrix}$$

Solve $Ax = e_1$ (solve $Ax = e_2$ solve $Ax = e_3$)

$x_i = \dfrac{\det A_i(e_1)}{\det A}$ $\left(x_i = \dfrac{\det A_i(e_2)}{\det A} \qquad x_i = \dfrac{\det A_i(e_3)}{\det A} \right)$

$$A^{-1} = \begin{bmatrix} \dfrac{\det A_1(e_1)}{\det A} & \dfrac{\det A_1(e_2)}{\det A} & \dfrac{\det A_1(e_3)}{\det A} \\ \dfrac{\det A_2(e_1)}{\det A} & \dfrac{\det A_2(e_2)}{\det A} & \dfrac{\det A_2(e_3)}{\det A} \\ \dfrac{\det A_3(e_1)}{\det A} & \dfrac{\det A_3(e_2)}{\det A} & \dfrac{\det A_3(e_3)}{\det A} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & -7 \\ 3 & 2 & 1 \end{bmatrix}$$

(1,1) position

$$\det A_1(e_1) = \det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & -7 \\ 0 & 2 & 1 \end{bmatrix} = (-1)^{1+1}\, 1 \det \begin{bmatrix} 5 & -7 \\ 2 & 1 \end{bmatrix}$$

(1,2)

$$\det A_2(e_1) = \det \begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & -7 \\ 3 & 0 & 1 \end{bmatrix} = (-1)^{1+2} \cdot 1 \cdot \det \begin{bmatrix} 2 & -7 \\ 3 & 1 \end{bmatrix}$$

(2,3) position

$$\det A_3(e_2) = \det \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 3 & 2 & 0 \end{bmatrix} = (-1)^{2+3} \cdot 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

(j,i) position

$$\det A_i(e_j) = \det \begin{bmatrix} & 1 & \\ & & \\ & & \end{bmatrix} = (-1)^{j+i} \cdot 1 \cdot \det A_{ji} = C_{ji}$$

recall

i-j cofactor $\quad C_{ij} = (-1)^{i+j} \det A_{ij}$

note indices are switched from what you'd expect here...

$$A^{-1} = \begin{bmatrix} \frac{\det A_1(e_1)}{\det A} & \frac{\det A_1(e_2)}{\det A} & \frac{\det A_1(e_3)}{\det A} \\ \frac{\det A_2(e_1)}{\det A} & \frac{\det A_2(e_2)}{\det A} & \frac{\det A_2(e_3)}{\det A} \\ \frac{\det A_3(e_1)}{\det A} & \frac{\det A_3(e_2)}{\det A} & \frac{\det A_3(e_3)}{\det A} \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Thus, we have this formula for the inverse,

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \tag{4}$$

The matrix of cofactors on the right side of (4) is called the **adjugate** (or **classical adjoint**) of A, denoted by adj A. (The term *adjoint* also has another meaning in advanced texts on linear transformations.) The next theorem simply restates (4).