Chapter 4

definitions

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

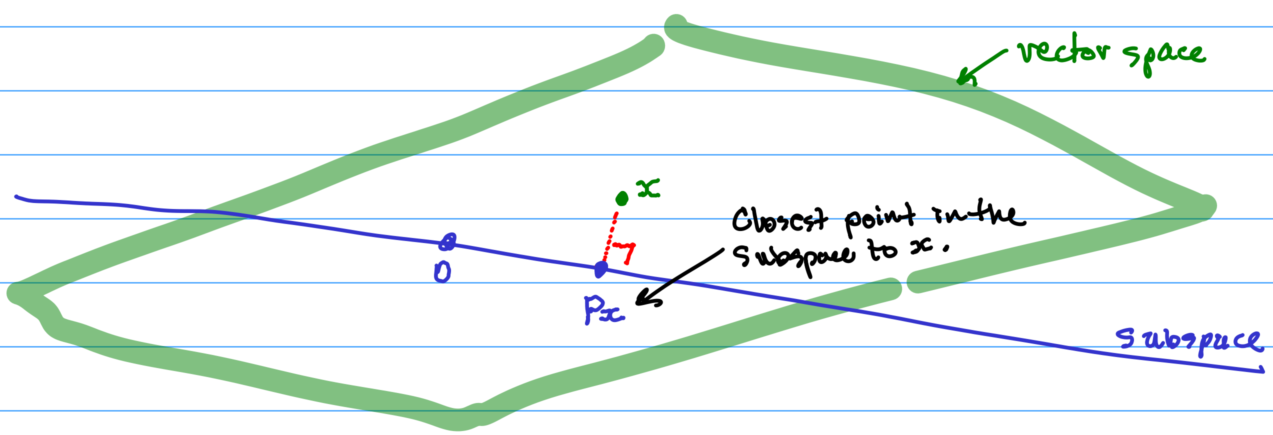
definition

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

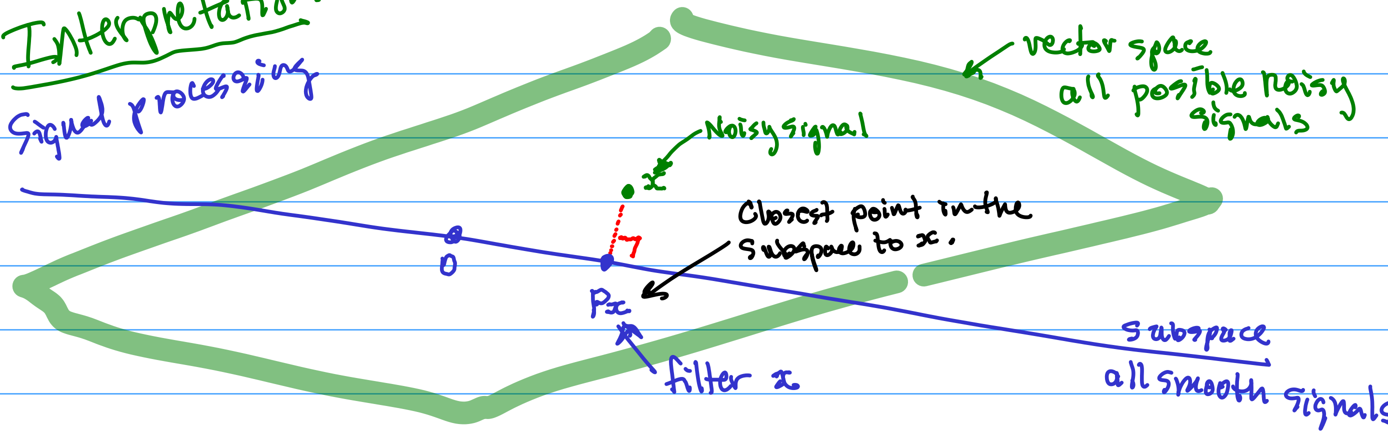
b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.



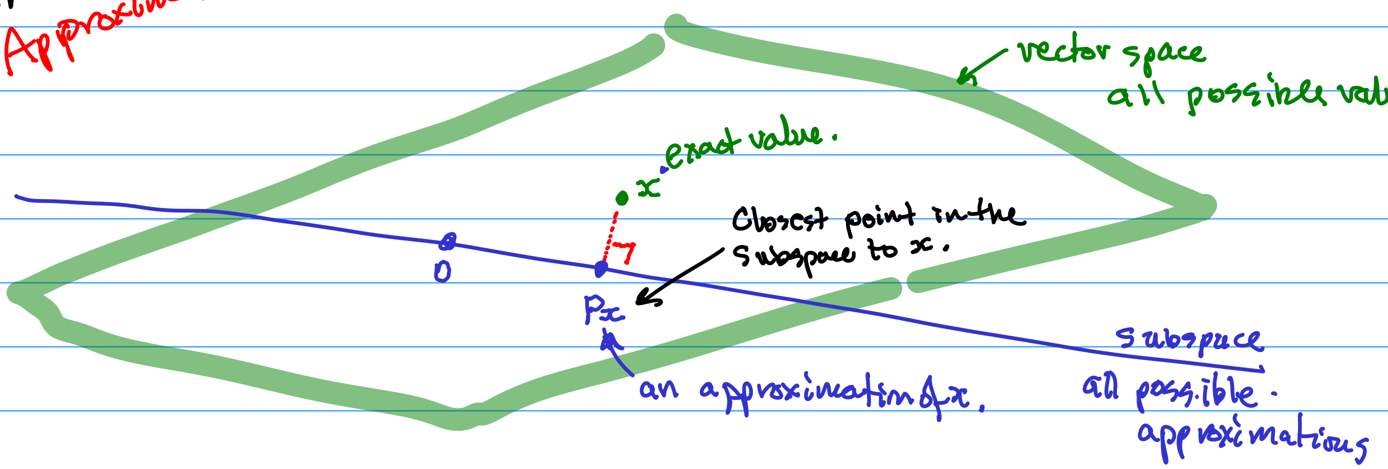
Interpretation

Signal processing



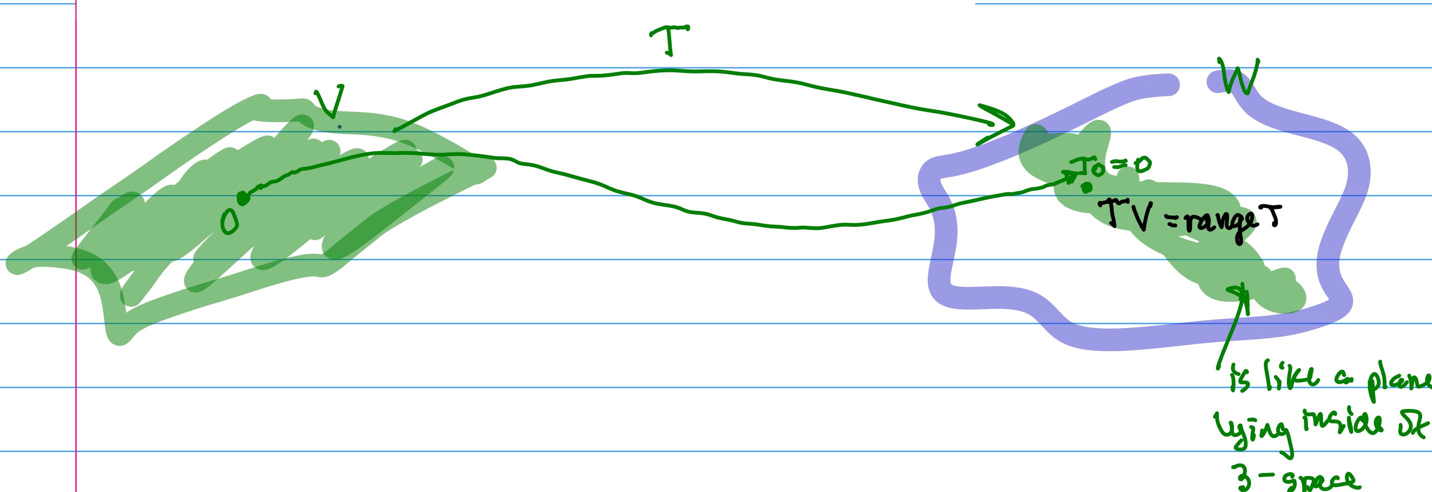
Interpretation

Approximation



Example of subspace

42. Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. [*Hint:* Typical elements of the range have the form $T(\mathbf{x})$ and $T(\mathbf{w})$ for some $\mathbf{x}$, $\mathbf{w}$ in V.]



A **subspace** of a vector space V is a subset H of V that has three properties:

range T

a. The zero vector of V is in H.[2] $\quad T\vec{0} = \vec{0}$ so $\vec{0} \in$ range T

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

(b) $u \in$ range T, $v \in$ range T need to show $u+v \in$ range T.

$u = T(\mathbf{x})$ for some $x \in V$

$v = T(\mathbf{w})$ for some $w \in V$

by linearity...

$u+v = T(\mathbf{x}) + T(\mathbf{w}) = T(x+w)$ thus $u+v \in$ range T

in the range of T

(c) $u \in$ range T, $c \in \mathbb{R}$ need to show $cu \in$ range T

$u = T(\mathbf{x})$ for some $x \in V$

$cu = cT(x) = T(cx)$ thus $cu \in$ range T.

by linearity. $\quad$ in the range

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

Same definition as before...

Find a nullspace by solving $Ax=0$ and writing the solution in parametrized vector form...

The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}}$$

already seen this

$$\text{Col } A = \{Ax : x \in \mathbb{R}^n\}.$$

Row space is the span of the rows...

$$\text{Row } A = \text{Col } A^T.$$

Examples:

1. Determine if $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}.$$

$$\text{Nul } A = \{x : Ax = 0\}$$

Need to check $Aw = 0$ ☑

$$\begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3 - 15 + 12 = 0$

$6 - 6 = 0$

$-8 + 12 - 4 = 0$

Example

echelon form

$$3.\quad A = \begin{array}{cccc} P & P & F & F \\ \end{array}$$
$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

Find Nul A and Col A ?

dim Nul A = # free variables = 2

rank A = dim Col A = # of pivots = 2

Since col $A \subseteq \mathbb{R}^2$ and dim col $A = 2$ then col $A = \mathbb{R}^2$.

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \qquad r_1 \leftarrow r_1 - 3r_2$$

reduced echelon form

$$\begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

$$x_1 - 7x_3 + 6x_4 = 0$$
$$x_2 + 4x_3 - 2x_4 = 0$$

$$x_1 = 7x_3 - 6x_4$$
$$x_2 = -4x_3 + 2x_4$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Nul A = Col N where $N = \begin{bmatrix} 7 & -6 \\ -4 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.