Basis of $H \subseteq \mathbb{R}^m$ set of vectors $B = \{b_1, b_2, \dots, b_n\} \subseteq \mathbb{R}^m$

① Independent

② Span the space H.

Means: If $x \in H$ then span means there are c_i's such that

$$x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$$

Define a linear operation that maps x into the c's.

$$[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Note if

$$A = \begin{bmatrix} b_1 \,|\, b_2 \,|\, \cdots \,|\, b_n \end{bmatrix} \in \mathbb{R}^{m \times n} \quad \text{then} \quad x = Ac$$

Questions: Given x is the c such that $x = Ac$ unique?

Note A is a matrix with lin. ind column. because the columns came from a basis.. Therefore there are no free variables. So there is only one c such that $Ac = x$.

This means the mapping $[x]_B = c$ is well defined.

Note linear is clear since $Ac = x$ implies the relationship between x and c is linear.

Like to solve for c:

$$Ac = x$$

like this $\cancel{c = A^{-1}x}$ ← this would have to work for all $x \in \mathbb{R}^m$

but is A invertible? No because A is not square unless $m = n$, which maybe it isn't.

That's why this notation...

$$c = [x]_B$$ for $x \in H$

The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

It says that if you know the dimension of a space and you have the right number of vectors, then

independence implies spanning

and/or,

spanning implies independence

Let's understand why independence implies spanning.

Let V be a p dimensional vector space...

$$V \subseteq \mathbb{R}^m \quad \text{and} \quad \dim V = p$$

more generally V could be a subspace of polynomials of a certain maximum degree. Or series of sine functions of a certain maximum frequency... etc...

The point of $\mathbb{R}^m$ is that the nature of the vectors in V is different than the dimension of the space...

Let $B = \{w_1, w_2, \ldots, w_p\} \subseteq \mathbb{R}^m$ be a basis of V. This basis exists and has p vectors in it because by hypothesis we already know $\dim V = p$.

To show independence implies spanning we consider another set $\{v_1, v_2, \ldots, v_p\}$ that is independent. and then see why these vectors must span V.

Since $v_1 \in V$ and B is a basis then there are c's such that

$$v_1 = c_1 w_1 + c_2 w_2 + \cdots + c_p w_p$$

$\boxed{[v_i]_B}$ can be found for every vector...

$$A = BC \quad \begin{cases} v_1 = w_1 c_{11} + w_2 c_{21} + \cdots + w_p c_{p1} \\ v_2 = w_1 c_{12} + w_2 c_{22} + \cdots + w_p c_{p2} \\ \vdots \\ v_p = w_1 c_{1p} + w_2 c_{2p} + \cdots + w_p c_{pp} \end{cases}$$

where

$$A = \left[v_1 \,\middle|\, v_2 \,\middle|\, \cdots \,\middle|\, v_p \right] \in \mathbb{R}^{m \times p} \qquad B = \left[w_1 \,\middle|\, w_2 \,\middle|\, \cdots \,\middle|\, w_p \right] \in \mathbb{R}^{m \times p}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & & \ddots & \\ c_{p1} & c_{p2} & \cdots & c_{pp} \end{bmatrix} \in \mathbb{R}^{p \times p}$$

even though we don't know what the vectors in V are really like, we can always write down C as a $p \times p$ matrix.

$A = BC$ where A has linearly independent columns.

Trying to show the columns of A span V.

Trying to solve $Ax = b$ for any $b \in V$.

That's the same as solving $BCx = b$.

That's the same as solving $\begin{cases} By = b \\ Cx = y. \end{cases}$

Since $B = \begin{bmatrix} w_1 & w_2 & \cdots & w_p \end{bmatrix}$ and the w's are a basis

then they span V and $By = b$ can be solved for every $b \in V$.

Can I solve $Cx = y$ for x?

Claim C is invertible.. It's square ☑.

Does it have free variables?

Does $Cx = 0$ have lots of solutions?

If $Cx = 0$ has lots of solution, then

$BCx = 0$ also has lots of solutions.. the same solutions as $Cx = 0$ and maybe some more..

Then $Ax = 0$ also has lots of solutions since $A = BC$.

by hypothesis the columns of A are independent, so A has no free variables, so the only solution to $Ax = 0$ is $x = 0$.

There $Cx = 0$ does not have lots of solutions... but only $x = 0$.

Therefore C is invertible and I can solve

$Cx = y$ as $y = C^{-1}x$.

Thus $Ax = b$ can be solved for every $b \in V$.

So

independence implies spanning.