In Exercises 7–10, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for $\mathbb{R}^2$. In each exercise, find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

7. $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$B = \left[b_1 \mid b_2 \right] = \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}$ $\qquad C = \left[c_1 \mid c_2 \right] = \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}$

$$[\mathcal{C} \leftarrow \mathcal{B}] = \left[[b_1]_{\mathcal{C}} \;\middle|\; [b_2]_{\mathcal{C}} \right]$$

Solve these equations

$[b_1]_{\mathcal{C}} = x$ means $\boxed{b_1 = x_1 c_1 + x_2 c_2 = C\,x}$

$[b_2]_{\mathcal{C}} = y$ means $\boxed{b_2 = y_1 c_1 + y_2 c_2 = C\,y}$

Solve $Cx = b_1$ and $Cy = b_2$ to find $[\mathcal{C} \leftarrow \mathcal{B}]$

Augmented matrix

$$\left[C \mid b_1 \, b_2 \right] = \left[C \mid B \right]$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right] \qquad r_2 \leftarrow r_2 + 5r_1$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ 0 & -8 & 40 & -16 \end{array}\right] \qquad r_2 \leftarrow \tfrac{-1}{8} r_2$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ 0 & 1 & -5 & 2 \end{array}\right] \quad r_1 \leftarrow r_1 + 2r_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$Cx = b_1$ and $Cy = b_2$

$x = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$ $\quad y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$[\mathcal{C} \leftarrow \mathcal{B}] = \left[[b_1]_{\mathcal{C}} \;\middle|\; [b_2]_{\mathcal{C}} \right] = [x \,|\, y] = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

$$[\mathcal{B} \leftarrow \mathcal{C}] = \left[[c_1]_{\mathcal{B}} \;\middle|\; [c_2]_{\mathcal{B}} \right]$$

To find $[c_1]_{\mathcal{B}}$, $[c_2]_{\mathcal{B}}$ use the augmented matrix.

$$[\,B \mid C\,]$$

Determine the dimensions of Nul A, Col A, and Row A for the matrices shown in Exercises 11–16.

$$\textbf{11.}\quad A = \begin{array}{c} \begin{array}{ccccc} P & P & F & P & F \end{array} \\ \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

already echelon form

$$\text{Col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 5 \\ 0 \end{bmatrix} \right\}$$

dim Col $A = 3$

Note A and A^T have the same # of pivots

dim Nul A = # of free variables = 2

dim Row A = dim Col A = 3

Determine the dimensions of Nul A, Col A, and Row A for the matrices shown in Exercises 11–16.

16. $A = \begin{array}{ccc} P & P & F \\ \hline \end{array}$
$$A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

dim Nul A = 1
dim Col A = 2
dim Row A = 2

space and A is an $m \times n$ matrix

Eigen Value – Eigen Vector Problem: Chapter 5...

Given $A \in \mathbb{R}^{n \times n}$

Solve $Ax = \lambda x$ for x and λ.

scalar

vector of length n

system of n equations

$n+1$ unknowns

Expect lots of solutions... because underdetermined

I want solutions $x \neq 0$ but $\lambda = 0$ is okay

note $x = 0$ satisfies $Ax = \lambda x$ for all λ, but this isn't useful.

Solve ... Collect terms

$$Ax - \lambda x = 0$$
$$Ax - \lambda I x = 0$$
$$(A - \lambda I)x = 0$$

note added the I so this factor makes sense...

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$ an upper triangular matrix

Consider the factor

$$A-\lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

When $(A-\lambda I)x = 0$ has solutions $x \neq 0$.

When $A-\lambda I$ has free variables then there are $x \neq 0$ that satisfy $(A-\lambda I)x=0$,

If $A-\lambda I$ has no free variabes the $x=0$ is the only solution.

$\lambda=6$

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} P & P & F \end{array} \\ \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Free variables

Thus $\begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} x = 0$ has lots of solutions such that $x \neq 0$.

These solutions are called the eigenvectors corresponding to the eigenvalue $\lambda = 6$

The free variable scales the length of the eigenvector. One could chose a specific eigen vectors and get the others by scaling it.

$\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} F & P & P \end{array} \\ \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \end{array}$$

again there are free variables so

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} x = 0$$

has solutions $x \neq 0$ which are the eigenvectors...

$\lambda = 4$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} P & F & P \end{array} \\ \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

$\lambda = 4$ is another eigenvalue...

Conclusion: The eigenvalues of an upper triangular matrix are the diagonal entries...