5.4 Eigenvectors and Linear Transformations

Basis $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ $\qquad P = [b_1 | b_2 | \cdots | b_n]$

$$W = \text{span}\,\mathcal{B} = \text{col}\,P$$

The $\mathcal{B}$-coordinate map...

Let $x \in W$. Then $[x]_{\mathcal{B}} = r$ means $x = Pr$

coordinates..

or $x = r_1 b_1 + r_2 b_2 + \cdots + r_n b_n$

axes

What is the coordinate map of a linear transformation?

$f(x) = Ax$ but $T(x) = Ax$

Find

use linearity T is linear and the coordinate map is also linear.

$$[T(x)]_{\mathcal{B}} = [T(r_1 b_1 + r_2 b_2 + \cdots + r_n b_n)]_{\mathcal{B}}$$

$$= \underbrace{r_1 [Tb_1]_{\mathcal{B}} + r_2 [Tb_2]_{\mathcal{B}} + \cdots + r_n [Tb_n]_{\mathcal{B}}}_{\text{identify this as a matrix multiplication.}}$$

$$= \underbrace{\Big[[Tb_1]_{\mathcal{B}} \Big| [Tb_2]_{\mathcal{B}} \Big| \cdots \Big| [Tb_n]_{\mathcal{B}} \Big]}_{\substack{\text{matrix} \\ [T]_{\mathcal{B}}}} \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

Thus

$$[T]_{\mathcal{B}} = \Big[[Tb_1]_{\mathcal{B}} \Big| [Tb_2]_{\mathcal{B}} \Big| \cdots \Big| [Tb_n]_{\mathcal{B}} \Big].$$

Let's suppose we've got a similarity $A = PDP^{-1}$

from last time

$A = PDP^{-1}$

$$[T]_\beta = \left[[T(b_1)]_\beta \,\middle|\, [T(b_2)]_\beta \,\middle|\, \cdots \,\middle|\, [T(b_n)]_\beta \right].$$

$$= \left[[Ab_1]_\beta \,\middle|\, [Ab_2]_\beta \,\middle|\, \cdots \,\middle|\, [Ab_n]_\beta \right].$$

$[Ab_1]_\beta = r_1$ means $Ab_1 = P r_1$ so $r_1 = P^{-1} A b_1$

$[Ab_1]_\beta = P^{-1}Ab_1$

recall

$[x]_\beta = r$ means $x = Pr$

$$[T]_\beta = \left[P^{-1}Ab_1 \,\middle|\, P^{-1}Ab_2 \,\middle|\, \cdots \,\middle|\, P^{-1}Ab_n \right]$$

now factor things out

$$= P^{-1}\left[Ab_1 \,\middle|\, Ab_2 \,\middle|\, \cdots \,\middle|\, Ab_n \right]$$

recall

$A = PDP^{-1}$

$$= P^{-1}A\left[b_1 | b_2 | \cdots b_n \right] = P^{-1}AP = P^{-1}PDP^{-1}P = D$$

A diagram...

$$\begin{tikzcd}
\mathbf{x} \arrow[rr, "T"] \arrow[d, "P^{-1}"] & & T(\mathbf{x}) \arrow[d, "P"'] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[rr, "\text{Multiplication}"', "\text{by } M=[T]_{\mathcal{B}}"] & & {[T(\mathbf{x})]_{\mathcal{B}}} \arrow[u, bend right, "A"']
\end{tikzcd}$$

5.5 Complex Eigenvalues

Suppose $A \in \mathbb{R}^{n \times n}$. Then the eigenvalues of A are the roots of the characteristic polynomial $\chi_A(\lambda) = \det(A - \lambda I)$.

Sometimes those roots are complex... even though the coefficients of the polynomial are real...

In $\mathbb{R}^2$ we have $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\chi_A(\lambda) = \det \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

Eigenvalues ... use quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad a=1, \quad b=-(a_{11}+a_{22})$$

$$c = a_{11}a_{22} - a_{12}a_{21}$$

when is $b^2 - 4ac < 0$?

$$(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$$

$$= (a_{11} + a_{22})^2 - 4a_{11}a_{22} + 4a_{12}a_{21} < 0$$

Claim: if $a_{12} = a_{21}$ then the above is never negative..

... next time...