Properties of symmetric Matrices.

☑ If $A = A^T$ then the eigenvalues (and eigenvectors) are real.

If $A = A^T$ then two eigenvectors of different eigenvalues are orthogonal.

Theorem 1 chapter 7.1

If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues and

$v_1 = Nul(A - \lambda_1 I)$ and $v_2 \in Nul(A - \lambda_2 I)$

Thus $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$

Claim $v_1 \cdot v_2 = 0$. Why?

symmetry $A^T = A$

$$\lambda_1 v_1 \cdot v_2 = A v_1 \cdot v_2 = (Av_1)^T v_2 = v_1^T A^T v_2 = v_1 \cdot A^T v_2$$

$$= v_1 \cdot A v_2 = v_1 \cdot \lambda_2 v_2 = \lambda_2 v_1 \cdot v_2$$

scalar

Therefore

$$\lambda_1 v_1 \cdot v_2 = \lambda_2 v_1 \cdot v_2$$

$$(\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$$

since $\lambda_1 \neq \lambda_2$ then $\lambda_1 - \lambda_2 \neq 0$ therefore $v_1 \cdot v_2 = 0$

We know that in general an nxn matrix has n eigenvalues and if it's a symmetric matrix the corresponding eigenvectors are orthogonal. Those vectors could be normalized to unit vectors.. This results in an orthonormal basis of eigenvectors for A.

The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d. A is orthogonally diagonalizable.

Spectral Theorem: If $A = A^T \in \mathbb{R}^{n\times n}$ then A has an orthonormal basis of eigenvectors and the eigenvalues are real. Namely there exists λ's and v's such that

$$\boxed{A v_i = \lambda_i v_i \quad \text{for } i = 1, \dots, n \qquad \text{with } \lambda_i \in \mathbb{R}.}$$

and

$$\boxed{v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}}$$

Put the eigenvectors in a matrix.

$$P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

columns of P are orthonormal vectors ... so $P^T P = I$

Also P is square so $P^{-1} = P^T$ and $PP^T = I$.

Then

$$AP = A\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}}_{D} = PD$$

Thus

$$AP = PD$$

$$A\cancel{P}\cancel{P^{-1}} = PDP^{-1}$$

$$A = PDP^{-1} = PDP^T \quad \text{or} \quad \boxed{A = PDP^T}$$

diagonal matrix

orthogonal matrices

7.4 Singular Value decomposition:

Recall an arbitrary matrix A may not be diagonalizable because there may not be a basis of eigenvectors... usually there is, but it's not always the case...

A symmetric matrix is always diagonalizable because the Spectral theorem says there is an orthonormal basis of eigenvectors... Always...

Idea. Given an arbitrary matrix that's not symmetric, create a related matrix that is and then use the spectral theorem.

Let $A \in \mathbb{R}^{m\times n}$ (not even square)

reflect. Note that $A = 0$ is a symmetric matrix. The only eigenvalue of the 0 matrix is 0, and any vector is an eigenvector... Thus, any orthogonal basis is a basis of eigenvectors for the zero matrix.

Note

$$B = \underset{n\times m}{A^T}\,\underset{m\times n}{A} \quad \text{then } B = B^T \in \mathbb{R}^{n\times n}$$

and

$$C = \underset{m\times n}{A}\,\underset{n\times m}{A^T} \quad \text{then } C = C^T \in \mathbb{R}^{m\times m}$$

At least B and C are square. Are they symmetric?

$$B^T = (A^TA)^T = A^T A^{TT} = A^TA = B$$

Now apply spectral theorem to B or to C.

Note, if $m > n$ then choose $B \in \mathbb{R}^{n \times n}$

if $n > m$ then choose $C \in \mathbb{R}^{m \times m}$

(if $m = n$ doesn't matter)

Let's just work with B.

By the the spectral theorem B has an orthonormal basis of eigenvectors.

$$Bu_i = \lambda_i u_i \text{ for } i = 1, \ldots, n \text{ and } u_i \cdot u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Let $U = [u_1 | u_2 | \cdots | u_n]$ note $U^T U = I$ and $UU^T = I$

Note, eigenvalues λ_i are real ...

$$\lambda_i u_i \cdot u_i = Bu_i \cdot u_i = A^T A u_i \cdot u_i = (A^T A u_i)^T u_i$$

$$= u_i^T A^T A^{TT} u_i = u_i^T A^T A u_i = A u_i \cdot A u_i$$

Thus

$$\lambda_i u_i \cdot u_i = A u_i \cdot A u_i$$

$$\lambda_i \|u_i\|^2 = \|A u_i\|^2 \qquad \text{so } \lambda_i \geq 0.$$

Relate the eigenvectors of $B = A^T A$ back to A

Define $y_i = A u_i$ and $z_i = \frac{y_i}{\|y_i\|} = \frac{y_i}{\sqrt{\lambda_i}}$

$$\|y_i\| = \|A u_i\| = \sqrt{\lambda_i}\, \|u_i\| = \sqrt{\lambda_i} \qquad y_i = \sqrt{\lambda_i}\, z_i$$

unit vectors

$$AU = A [u_1 | u_2 | \cdots | u_n] = [Au_1 | Au_2 | \cdots | Au_n]$$

$$= \left[y_1 \middle| y_2 \middle| \cdots \middle| y_n \right] = \left[\sqrt{\lambda_1} z_1 \middle| \sqrt{\lambda_2} z_2 \middle| \cdots \middle| \sqrt{\lambda_n} z_n \right]$$

$$= \underbrace{\left[z_1 \middle| z_2 \middle| \cdots \middle| z_n \right]}_{V} \underbrace{\begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \quad \sqrt{\lambda_n} \end{bmatrix}}_{\Sigma} = V\Sigma$$

$AU = V\Sigma$ or $\boxed{A = V\Sigma U^T}$

singular value decomposition of A.