5. Let A be the matrix and x be the vector given by

$$A = \begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} \qquad \text{and} \qquad x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Show that x is an eigenvector of A and find the eigenvalue.

Warning don't try to find the eigenvalues using the characteristic polynomial

$$\chi_A(\lambda) = \det(A - \lambda I)$$

because we already know an eigenvector...

Instead use the fact that $Ax = \lambda x$ and solve for λ since we already know x.

$$\begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-12+7 \\ 3\cdot -4+7 \\ 5-12+4 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ ? \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Thus $\lambda = -3$

(iv) $\det(A^{-1}) = 1/\det(A)$.

(A) True

(B) False

why: equivalent to $\det(A)\det(A^{-1}) = 1$

$\det(AA^{-1}) = 1$

$\det I = 1$

not a precise proof, but enough to prevent me from just guessing...

$I = 1\!\!1$

1 is the mult. identity for regular mult.

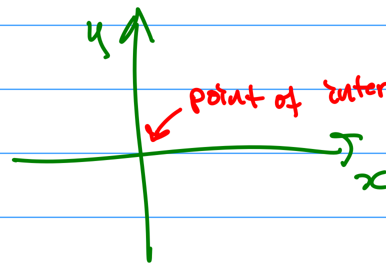
I is the mult. identity for matrix mult.

(vi) If W is a subspace of $\mathbf{R}^n$ and v is in both W and $W^{\perp}$, then $v = 0$.

(A) True

(B) False

Why? idea... W is like x-axis
$W^{\perp}$ is like y-axis



(vii) If $A = QR$ where Q has orthonormal columns, then $R = Q^T A$.

(A) True

(B) False

why $Q^TQ = I$ so

$$A = QR$$
$$Q^TA = Q^TQR = R$$

(ix) Every matrix $A \in \mathbf{R}^{n\times n}$ can be factored as $A = SDS^{-1}$ where D is diagonal and S is an invertible matrix.

(A) True

(B) False

Why some matrices don't have a basis of eigenvectors.

why: Example $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$\chi_A(\lambda) = \det\begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 = 0$$

So eigenvalues are $\lambda = 2$ (with mult 2.)

Question... find the eigenvectors and do the eigenvector form a basis of $\mathbb{R}^2$?

$\lambda = 2$

$$\mathrm{Nul}(A - \lambda I) = \mathrm{Nul}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \left\{ x : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0 \right\}$$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$ means $x_2 = 0$
$x_1 = \text{free}$.

Nul is 1 dimensional

solutions $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

only one vector...

8. Suppose $A \in \mathbf{R}^{2\times 2}$ is given by

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix}.$$

Use the Gram-Schmidt algorithm to factor $A = QR$ where Q is a matrix with orthonormal columns and R is upper triangular.

$$Q = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{10} & -7/\sqrt{10} \\ 0 & 9/\sqrt{10} \end{bmatrix}$$

check columns are orthonormal $\boxed{\checkmark}$

$$z_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \|z_1\| = \sqrt{1+9} = \sqrt{10} \qquad q_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$z_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \left(\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \left(\frac{1}{\sqrt{10}} (-7) \right) \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \frac{7}{10} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 27/10 \\ 9/10 \end{bmatrix} = \frac{9}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\|z_2\| = \frac{9}{10}\sqrt{9+1} = \frac{9}{10}\sqrt{10} = \frac{9}{\sqrt{10}}$$

$$\frac{z_2}{\|z_2\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

10. The LU factorization of a matrix A is given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/3 & -2/3 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Explain how to use this factorization to solve the equation $Ax = b$ and then find the value of x corresponding to $b = (4, 6, 17)$.

Warning: Do NOT multiply LU together to find A and then try to solve $Ax=b$

$Ax = b$ know $A = LU$

$LUx = b$

substitute $y = Ux$ to get $Ly = b$

Solve the system of systems.

$$\begin{cases} Ly = b & \leftarrow \text{by substitution} \\ Ux = y & \leftarrow \text{by substitution} \end{cases}$$

11. The QR factorization of a matrix A is given by

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{4}{3\sqrt{5}} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 3 & \frac{1}{3} \\ 0 & \frac{4\sqrt{5}}{3} \end{bmatrix}.$$

don't mult QR together to find A and then solve $A^TAx = A^Tb$

Explain how to use this factorization to minimize $\|Ax - b\|$ and then find the minimizing value of x corresponding to $b = (1, 0, 1)$.

Solve $Rx = Q^Tb$ to find the minimizer..

by substitution..