- Section 2.1 # 28

28 Suppose the second column of B is all zeros. What can you say about the second column of AB?

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$.

Since

$$AB = A\left[\, b_1 \;\middle|\; \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \;\middle|\; b_3 \;\middle|\; \cdots \;\middle|\; b_p \,\right] = \left[\, Ab_1 \;\middle|\; A0 \;\middle|\; Ab_3 \;\middle|\; \cdots \;\middle|\; Ab_p \,\right]$$

$$= \left[\, Ab_1 \;\middle|\; \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \;\middle|\; Ab_3 \;\middle|\; \cdots \;\middle|\; Ab_p \,\right]$$

one can say the second column of AB is also zero,

- Section 2.2 # 23

23. Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?

If A is invertible then A^{-1} exists. It follows after multiplying by A^{-1} on the left that

$$A^{-1}(AB) = A^{-1}(AC)$$

Then by associativity of matrix-matrix multiplication

$$(A^{-1}A)B = (A^{-1}A)C$$

or

$$IB = IC$$

and therefore

$$B = C.$$

This is not true in general. Suppose $B \neq C$ and that $A = 0$ is the zero matrix. Then

$$AB = 0B = 0$$

and

$$AC = 0C = 0$$

so $AB = AC$. However $B \neq C$ as assumed.

- Section 2.2 # 31

31. Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.

If A is invertible then $Ax=0$ has only one solution, the trivial solution $x=0$.

On the other hand, if the columns of A were not linearly independent there would be a dependency relation between the columns of A given as

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$$

where $x \neq 0$ and the $a_i \in \mathbb{R}^n$ are the columns of A.

But then the column representation of matrix-vector multiplication would imply

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$$

or, in particular, that $Ax=0$. Since $x \neq 0$ this would contradict $Ax=0$ having only the trivial solution. Therefore the columns of A are linearly independent.