- Section 5.1 # 31

31. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

Recall Theorem 2 from Section 5.1 in the book

> If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

If there were more than n distinct eigenvalues there would be more than n corresponding eigenvectors. Since the theorem says the eigenvectors corresponding to distinct eigenvalues are linearly independent that would imply there is a set of linearly dependent vectors in $\mathbb{R}^n$ with more than n vectors in it. As that in an impossibility, then an $n \times n$ matrix can have at most n distinct eigenvalues.

Alternative Solution

This problem could also be done using the characteristic polynomial. Let $A \in \mathbb{R}^{n \times n}$. Then the characteristic polynomial

$$\chi_A(\lambda) = \det(A - \lambda I)$$

is a polynomial of degree n. Therefore $\chi_A(\lambda) = 0$ has at most n roots. As the roots of the characteristic polynomial correspond to the eigenvalues of A, there are at most n eigenvalues.

- Section 5.2 # 20

20. Use a property of determinants to show that A and A^T have the same characteristic polynomial.

Since $(A-\lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$ and the determinant is unchanged by taking transposes, then

$$\chi_A(\lambda) = \det(A-\lambda I) = \det((A-\lambda I)^T) = \det(A^T-\lambda I) = \chi_{A^T}(\lambda)$$

Therefore, A and A^T have the same characteristic polynomial.

- Section 5.3 # 33

33. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

If A is diagonalizable then $A = PDP^{-1}$ where P is invertible and D is a diagonal matrix. It follows that $D = P^{-1}AP$ and since all the matrices on the right are invertible, then D on the left is also invertible. In fact

$$D^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P$$

Rearranging this yields that

$$A^{-1} = PD^{-1}P^{-1}$$

and so A^{-1} is similar to D^{-1}. To finish the argument not that

Since D is diagonal then

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \quad \text{and} \quad D^{-1} = \begin{bmatrix} 1/\lambda_1 & & & 0 \\ & 1/\lambda_2 & & \\ & & \ddots & \\ 0 & & & 1/\lambda_n \end{bmatrix}$$

shows D^{-1} is also diagonal. Therefore A^{-1} is diagonalizable.

Remark, before writing down the inverse of D is was necessary to show that D is actually invertible. Otherwise one of the λ's might be zero and the inverse not exist,