ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1] $r_i \leftarrow r_i + \alpha r_j$ $i \neq j$ add α times row j to row i
2. (Interchange) Interchange two rows. $r_i \leftrightarrow r_j$ $i \neq j$ switch row i with row j
3. ~~(Scaling) Multiply all entries in a row by a nonzero constant.~~ $r_i \leftarrow \alpha r_i$ $\alpha \neq 0$ rescale row i by α.

$$\begin{cases} x_1 - 2x_2 - x_3 = 3 \\ 3x_1 - 6x_2 - 2x_3 = 2 \end{cases}$$

$$Ax = b$$

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -6 & -2 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Augmented matrix

$$\underbrace{[A \,\vdots\, b]}_{\text{partioned matrix}} = \left[\begin{array}{ccc:c} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{array}\right]$$

Find row echelon form working from left column to the right

choose 1 as the pivot

$$r_2 \leftarrow r_2 - \tfrac{3}{1} r_1$$

Write the row operations carefully because we'll use them later for making matrix factorizations.

$$\left[\begin{array}{ccc:c} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

Pivots

Echelon form because zeros in the lower left triangle...

Find reduced echelon form:

$$r_1 \leftarrow r_1 + r_2$$

$$\begin{array}{cccc} P & F & P & \\ x_1 & x_2 & x_3 & b \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & -7 \end{array}\right]$$

Convert back to algebraic form

$$x_1 - 2x_2 = -4$$
$$x_3 = -7$$

Free variable

$$x_1 - \cancel{2x_2} = -4$$
$$x_3 = -7$$

The free variable could be anything

$x_1 = -4$

$x_3 = -7$

solution

$$x = \begin{bmatrix} -4 \\ 0 \\ -7 \end{bmatrix}$$

$$x_1 = -4 + 2x_2$$
$$x_3 = -7$$

Free variable is equal to itself.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4+2x_2 \\ x_2 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 0 \\ -7 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -7 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{} x_2$$

free variable

all the vectors of this form is called the free space... or null space of A

- This is the case where there were an infinite # of solutions

- The free space depends on the matrix A but not b.

$$N = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -6 & -2 \end{bmatrix}$$

$$AN = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -6 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 - x_3 \\ 3x_1 - 6x_2 - 2x_3 \end{bmatrix} = \begin{bmatrix} 2-2-0 \\ 6-6-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(x_1, x_2, x_3) = (2, 1, 0)$

Another Example

$$\begin{aligned} 1x_1 - 7x_2 + 6x_4 &= 5 \\ x_3 - 2x_4 &= -3 \\ -x_1 + 7x_2 - 4x_3 + 2x_4 &= 7 \end{aligned}$$

$$A = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$A = \begin{bmatrix} 1 & -7 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ -1 & 7 & -4 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$$

Augmented matrix

pivot

$$[A \,\vdots\, b] = \left[\begin{array}{cccc:c} \boxed{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right] \qquad r_3 \leftarrow r_3 + r_1$$

next pivot

$$\left[\begin{array}{cccc:c} \boxed{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & \boxed{1} & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right] \qquad r_3 \leftarrow r_3 + 4r_2$$

$$\begin{array}{cccc} P & F & P & F \end{array}$$
$$\left[\begin{array}{cccc:c} \boxed{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & \boxed{1} & -2 & -3 \\ \underbrace{0 \quad 0 \quad 0 \quad 0}_{0 \text{ on the left}} & & & & \underbrace{0}_{0 \text{ on the right}} \end{array}\right]$$

echelon form... (actully already reduced)

if not a zero here then there would be no solution...

Solve for the pivot variables in terms of the free...

$$x_1 - 7x_2 + 6x_4 = 5 \qquad x_1 = 5 + 7x_2 - 6x_4$$
$$x_3 - 2x_4 = -3 \qquad x_3 = -3 + 2x_4$$

Solution

Free variables

$$x = \begin{bmatrix} 5 + 7x_2 - 6x_4 \\ x_2 \\ -3 + 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \end{bmatrix} x_4}_{\text{null space of } A.}$$