A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a linear tranformation if

- $T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$
- $T(cu) = cT(u)$ for all $c \in \mathbb{R}$ and $u \in \mathbb{R}^n$.

Claim there exists a matrix $A \in \mathbb{R}^{m \times n}$ such that $T(x) = Ax$.

Standard basis of $\mathbb{R}^n$ .. let $x \in \mathbb{R}^n$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

where $e_i \in \mathbb{R}^n$ is the vector with all all zero entries except the ith entry which is 1.

Thus

$$T(x) = T(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) = T(x_1 e_1) + T(x_2 e_2) + \cdots T(x_n e_n)$$

$$= \underline{x_1} T(e_1) + \underline{x_2} T(e_2) + \cdots + \underline{x_n} T(e_n)$$

What is Ae_1? Example

$$Ae_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}(1) + \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}(0) + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}(0) = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$$Ae_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}(0) + \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}(1) + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}(0) = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$T(x) = T(x_1e_1 + x_2e_2 + \cdots + x_ne_n) = T(x_1e_1) + T(x_2e_2) + \cdots T(x_ne_n)$$

$$= x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n)$$

$$= \begin{bmatrix} \vdots & \vdots & & \vdots \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Therefore.

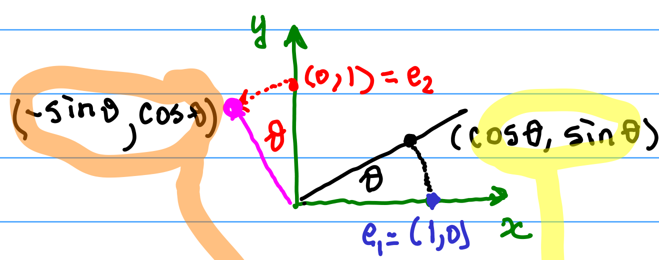
$T(x) = Ax$ where $A = \begin{bmatrix} \vdots & \vdots & & \vdots \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ \vdots & \vdots & & \vdots \end{bmatrix}$

Conclusion: All linear transformations can be seen as matrix-vector multiplication... What's interesting is the two properties

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$

is all that's needed for this to work.

Example: Rotation by θ



definition of $\cos\theta$ and $\sin\theta$ are the points on the unit circle.

$$R = \begin{bmatrix} Re_1 & \vdots & Re_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

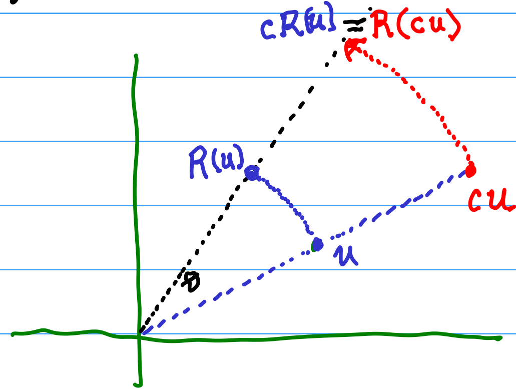
provided rotation is a linear transformation

Is rotation linear? Need to show

$R(u+v) = R(u) + R(v)$ also $R(cu) = c\,R(u)$

Claim $R(cu) = c\,R(u)$

just a picture



Claim $R(u+v) = R(u) + R(v)$

think about this at home...

Row operations are linear transformations...

Elimination step

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad r_1 \leftarrow r_1 + 2r_2$$

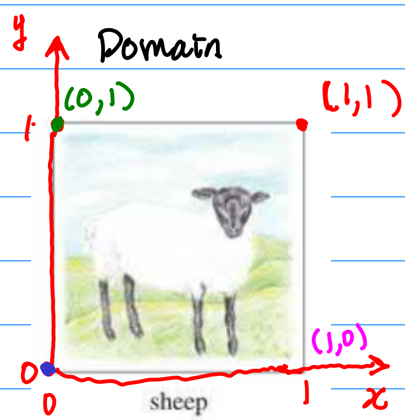
$T(0,1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

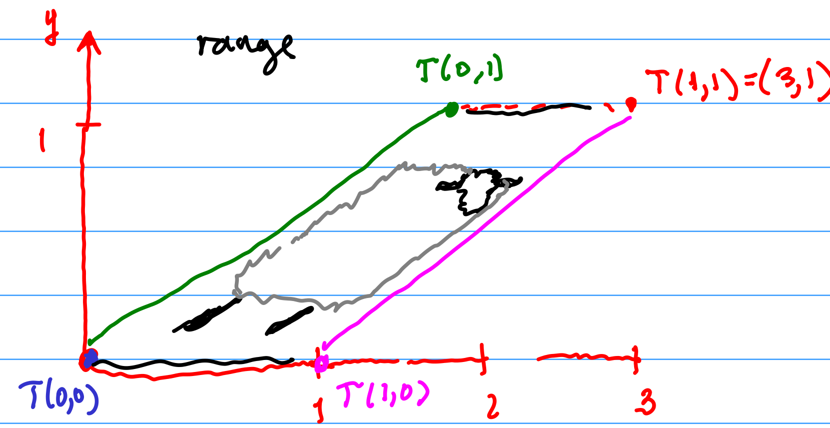
$T(1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \leftarrow$$

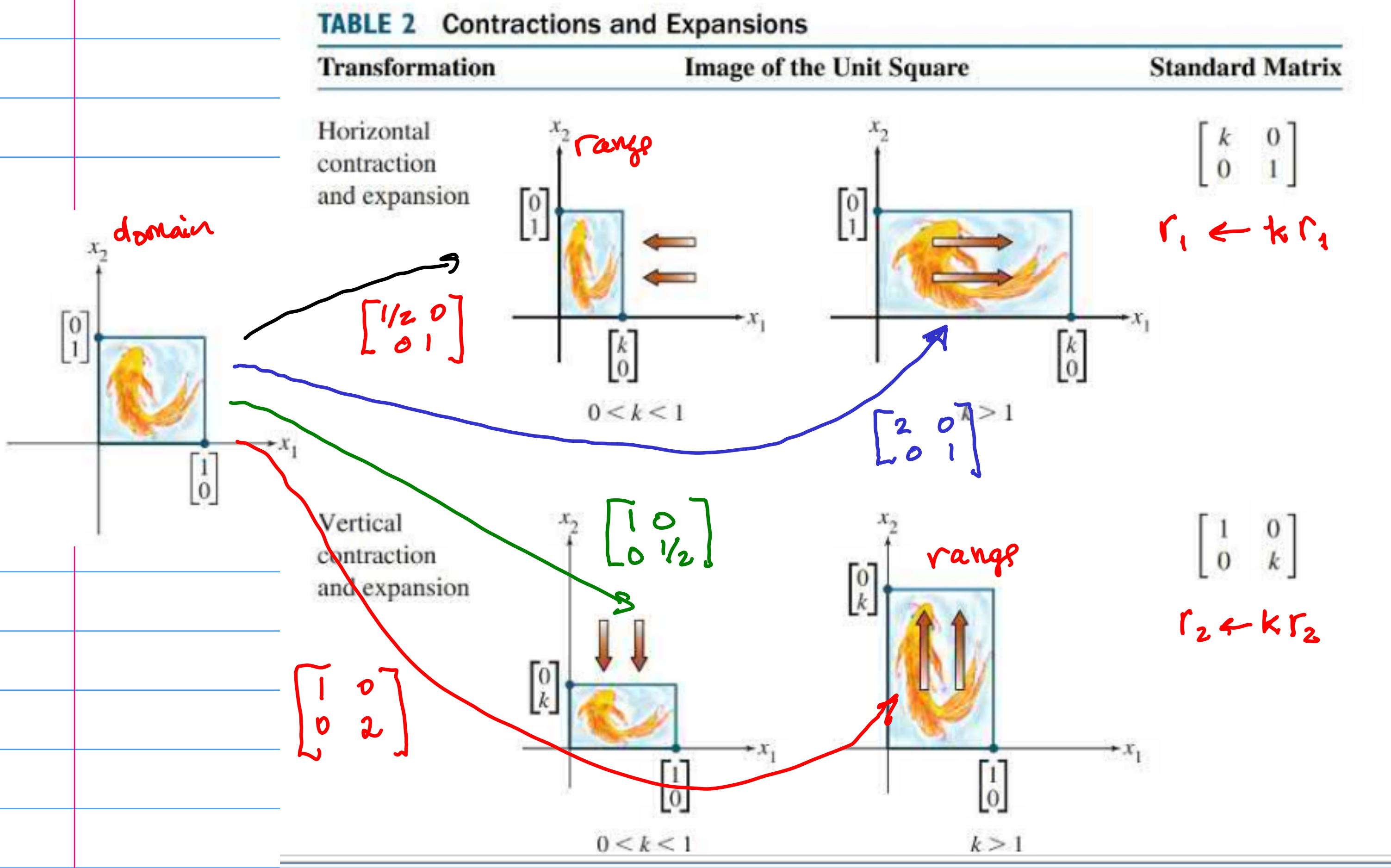
Matrix corresponding to elimination step...

$$T(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ y \end{bmatrix}$$





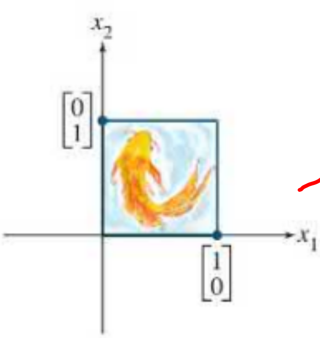
Scaling row operation...



Elimination row swap

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad r_1 \leftrightarrow r_2$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Reflection through the line $x_2 = x_1$

$$T(x,y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

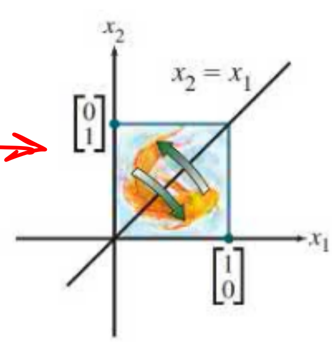
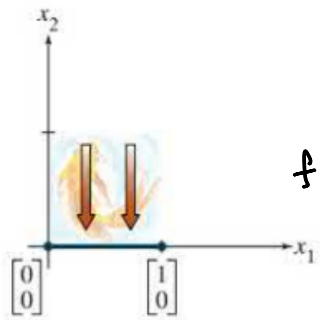
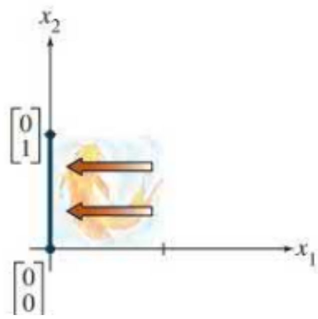


TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

not scaling by 0 because that's not allowed...

flattening the fish onto the x-axis

instead call it a projection

Why is scaling by zero to a elementary row operation?

Because that's not invertible. That is ...
you cant divide by zero to undo a projection.