Inverse of a Matrix

$A \in \mathbb{R}^{m \times n}$ ← rows (m), columns (n)

$f: \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = Ax$ ← since f is given by a matrix it's linear.

Inverse function: undo what f does. $(g \circ f)(x) = x$

the domain of g is the range of f

Claim g is a linear function: Need to show

- $\boxed{\checkmark}$ $g(u+v) = g(u) + g(v)$
- $g(cu) = c\,g(u)$

Let $u = f(x)$ ← since u is in the domain of g it must be in the range of f so

$v = f(y)$

$g(u) = g(f(x)) = (g \circ f)(x) = x$ $\qquad g(v) = y$

$u + v = f(x) + f(y) = f(x+y)$ since f is linear

$\boxed{\checkmark}$ $g(u+v) = \underbrace{(g \circ f)}_{\text{inverse}}(x+y) = x + y = g(u) + g(v)$.

$cu = c\,f(x) = f(cx)$ since f is linear

$g(cu) = \underbrace{g(f(cx))}_{\text{inverse}} = cx = c\,g(u)$

Since g is linear we can look for a matrix that corresponds to g.

$f(x) = y$ try to map y back to x

$Ax = y$ solve for x given any y.

Algorithm: Make the echelon form of A. If there is a pivot in every row then we can solve for x.

Guaranteed this can't be done if there are more rows in A than columns. Since a pivot in every row means at least that many columns then

$A \in \mathbb{R}^{m \times n}$ rows columns means $m > n$ implies there is no inverse.

Idea to find the inverse. (what about if $m < n$?)

$y \in \mathbb{R}^m \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = y_1 e_1 + y_2 e_2 + \cdots + y_m e_m$

... where $e_i \in \mathbb{R}^m$ that is 0 in every component except the ith component which is 1.

assume a pivot in every rows so $m \le n$

Solve $Av_1 = e_1$, $Av_2 = e_2$, ..., $Av_m = e_m$ for v's.

here $v_1 \in \mathbb{R}^n$ $\quad v_2 \in \mathbb{R}^n$ ... $\quad v_m \in \mathbb{R}^n$

$$y = y_1 e_1 + y_2 e_2 + \cdots + y_m e_m = y_1 A v_1 + y_2 A v_2 + \cdots + y_m A v_m$$

$$\underbrace{A(y_1 v_1) + A(y_2 v_2) + \cdots + A(y_m v_m)}_{\text{linearity}} \underset{\text{linearity}}{=} A(\underbrace{y_1 v_1 + y_2 v_2 + \cdots + y_m v_m}_{x})$$

Let $x = \underbrace{y_1 v_1 + y_2 v_2 + \cdots + y_m v_m}_{\text{matrix mult}}$ then $Ax = y$

column view of matrix vector mult

$$= \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}}_{A^{-1} \text{ inverse matrix}} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$x = g(y) = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} y = A^{-1} y$$

$n \times m$ — rows, cols. where $m \le n$

at this point $x = A^{-1}y$ is the solution to $Ax = y$ so we could plug it in to obtain $\underset{m\times n}{A}\underset{n\times m}{A^{-1}}y = y$ for every $y \in \mathbb{R}^m$.

We also want $A^{-1}Ax = x$ for every $x \in \mathbb{R}^n$. For that try

$A^{-1}y = x$ and think about solving for $y \in \mathbb{R}^n$ in terms of $x \in \mathbb{R}^n$.

If this can be solved for every x then there must be a pivot in every row of the echelon form of A^{-1},

this means A^{-1} can't have more rows than columns

.... thus $n > m$ is impossible.

Thus $m = n$ and A must be square.

Solve $Av_1 = e_1$, $Av_2 = e_2$, ..., $Av_m = e_m$ for v's.

Use augmented matrix to solve these systems

$[A \mid e_1]$ $\quad$ $[A \mid e_2]$ $\quad \dots \quad$ $[A \mid e_m]$

Since A is square and there is a pivot in every row

$\left[\underbrace{\text{reduced Echelon form of } A}_{I} \,\middle|\, v_1\right]$ $\quad$ $\left[\underbrace{\text{reduced Echelon form of } A}_{I} \,\middle|\, v_2\right]$ $\quad$ $\left[\underbrace{\text{reduced Echelon form of } A}_{I} \,\middle|\, v_m\right]$

Since the row operation are the same no matter what the right side of the system is, we can make a larger augmented matrix and do the row operations only once

$$\Big[\underset{m\times n}{A} \,\Big|\, \underbrace{e_1 \,\vdots\, e_2 \,\vdots\, \cdots \,\vdots\, e_m}_{I \text{ since square}}\Big] \xrightarrow{\text{row op}} \Big[\, I \,\Big|\, \underbrace{v_1 \,\vdots\, v_2 \,\vdots\, \cdots \,\vdots\, v_m}_{\text{that's } A^{-1}}\Big]$$

Recall... $m=n$ ... so $A \in \mathbb{R}^{n \times n}$ and $A^{-1} \in \mathbb{R}^{n \times n}$ are square

If $Ax = y$ we can get x back from y by $x = A^{-1}y$

Also multiplying both sides by A^{-1} yields

$$A^{-1}Ax = A^{-1}y = x \qquad \text{so} \qquad A^{-1}Ax = x \quad \text{for all } x \in \mathbb{R}^n$$

Similarly mult by A on both sides

$$y = Ax = AA^{-1}y \qquad \text{for all } y \in \mathbb{R}^n$$

In the end $\quad A^{-1}A = I \quad$ and $\quad AA^{-1} = I$.

Example: $n=2$, $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\left[\,A \;\middle|\; e_1 \,\vdots\, e_2\,\right] = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \qquad r_2 \leftarrow r_2 - 3r_1$$

echelon $\quad \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \qquad r_1 \leftarrow r_1 + r_2$

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \qquad r_2 \leftarrow -\tfrac{1}{2} r_2$$

$$\left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

Therefore. $\quad A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$