$$
\begin{array}{cccc}
 & p & & \\
A = & \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}
\end{array}
$$

$r_2 \leftarrow r_2 + r_1$

$r_3 \leftarrow r_3 - 2r_1$

row col

put 2 in the 3,2 slot

$$
\begin{array}{ccc}
 & p & \\
\end{array}
\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix}
$$

$r_3 \leftarrow r_3 + 5r_2$

$$U = \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

Having factored $A = LU$ into two simpler matrices, we can solve $Ax = b$ by two simpler systems $Ly = b$ and then $Ux = y$.

In this case solving $Ly = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$ gave $y = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$

Now solve $Ux = y$

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$$ for $x = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$

Write out using usual algebraic notation

$$3x_1 - 7x_2 - 2x_3 = -7$$
$$-2x_2 - 1x_2 = -2$$
$$-1x_3 = 6$$

"back" substitution

start with last variable

$x_3 = -6$

## Another Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 2 & 5 & 2 \end{bmatrix}$$

$r_2 \leftarrow r_2 - r_1$

$r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 1 & -4 \end{bmatrix}$$

can't always make LU factorization

oops... there is a zero in the pivot position

just switch rows.

permuted LU factorization PLU

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

already the echelon form...

What to do with rowswap.

$$\underbrace{[r_2 \longleftrightarrow r_3]}^{\text{3rd step}} \underbrace{[r_3 \leftarrow r_3 - 2r_1]}^{\text{2nd step}} \underbrace{[r_2 \leftarrow r_2 - r_1]}^{\text{1st step}} A = U$$

$$A = \underbrace{[r_2 \leftarrow r_2 + r_1][r_3 \leftarrow r_3 + 2r_1]}_{\text{like } L} \underbrace{[r_2 \longleftrightarrow r_3]}_{\text{what's this.}} U$$

working with the new $r_3$ is the same as the original $r_2$.

$$A = [r_2 \leftarrow r_2 + r_1] [r_2 \longleftrightarrow r_3] [r_2 \leftarrow r_2 + 2r_1] U$$

$$A = [r_2 \longleftrightarrow r_3] [r_3 \leftarrow r_3 + r_1] [r_2 \leftarrow r_2 + 2r_1] U$$

no matter how many row swap and elimination steps, you can always rewrite them so the row swaps are first the the elimations steps all together..

$$A = \overbrace{[r_2 \leftrightarrow r_3]}^{P}\ \overbrace{[r_3 \leftarrow r_3 + r_1]\ [r_2 \leftarrow r_2 + 2r_1]}^{L}\ U$$

row col

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P = [r_2 \leftrightarrow r_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore $A = PLU$ permuted LU factorization...

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

How to use this?

$$Ax = b$$

$$P\underbrace{LU}_{y}x = b$$

$$PLy = b$$

$$Ly = P^{-1}b$$

note the columns of P are unit vectors which are perpendicular to each other... square matrices whose columns are unit vectors that are also perpendicular to each other are called orthogonal matrices...

$$\begin{cases} Ly = P^{-1}b \\ Ux = y \end{cases}$$

now solve the system of systems

Permutation matrices have a geometric meaning, they just swap the axis around.. Remark $P^{-1} = P^T$ always the case...

## 2.8 Subspaces of $\mathbb{R}^n$

A **subspace** of $\mathbb{R}^n$ is any set $H$ in $\mathbb{R}^n$ that has three properties:

- ☑ a. The zero vector is in $H$. $0 \in H$
- ☑ b. For each $\mathbf{u}$ and $\mathbf{v}$ in $H$, the sum $\mathbf{u}+\mathbf{v}$ is in $H$. $u, v \in H$ implies $u+v \in H$
- ☐ c. For each $\mathbf{u}$ in $H$ and each scalar $c$, the vector $c\mathbf{u}$ is in $H$.

$u \in H$, $c \in \mathbb{R}$ implies $cu \in H$.

---

Let $A \in \mathbb{R}^{m \times n}$ $\quad f(x) = Ax$, $\quad f: \mathbb{R}^n \to \mathbb{R}^m$

soln. to homogeneous problem

$$\text{Nul } A = \{ x \in \mathbb{R}^n : Ax = 0 \} \subseteq \mathbb{R}^n$$

$$\text{Col } A = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$

range of the function $f(x) = Ax$

We'll talk about basis for a subspace next week. Now check these are subspaces.

Check Nul A is a subspace: is $0 \in \text{Nul } A$

is $A0 = 0$ Yes!

Suppose $u, v \in \text{Nul } A$. Then $Au = 0$ and $Av = 0$

since matrix-vector mult is a linear func.

then $0 + 0 = Au + Av = A(u+v) = 0$

property c also uses the linearity.