Let $\quad C \in \mathbb{R}^{q \times p} \qquad D \in \mathbb{R}^{p \times q}$

$\underset{q\times p \;\; p\times q}{CD} \in \mathbb{R}^{q\times q} \qquad \underset{p\times q \;\; q\times p}{DC} \in \mathbb{R}^{p\times p}$

are both square matrices.

$\begin{bmatrix} & D & \end{bmatrix}$

columns of D are the same as # columns of CD

$\begin{bmatrix} C \end{bmatrix} \begin{bmatrix} CD \end{bmatrix}$

# of rows of C are the same as # of rows of CD.

Theorem Suppose $C \in \mathbb{R}^{q\times p}$, $D \in \mathbb{R}^{p\times q}$ and $CD = I$ and $\boxed{DC = I}$. Then $p = q$, that is, the original matrices must have been square.

Suppose $q < p$. Then C has fewer rows than columns. i.e. the corresponding linear system has more variables than equations.. Thus C has free variables. So there is a non-zero vector x such that $Cx = 0$, That is Nul C has lots of non-zero vectors it in.

Consequently

$$x = Ix = (DC)x = D(Cx) = D0 = 0$$

but x was taken to be non-zero.. Something is wrong.
Therefore (not $q < p$) or in other words $q \geq p$.
Since also $CD = I$ by symmetry we conclude $q = p$.

A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

If $\mathcal{B} = \{u_1, u_2, \ldots, u_p\}$ is a basis then

$x \in H$ implies there are $c_i \in \mathbb{R}$ such that

$$x = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p$$

or

$$x = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix}}_{A \in \mathbb{R}^{n \times p}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}}_{c \in \mathbb{R}^p}$$

spans H

$$H = \operatorname{Col} A = \{Ac : c \in \mathbb{R}^p\}$$

linearly independent means

$$0 = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p \quad \text{only if } c_i = 0 \text{ for all } i = 1, \ldots, p$$

or $\quad Ac = 0$ only when $c = 0$

or $\quad \operatorname{Nul} A = \{x : Ax = 0\} = \{0\}$.

In summary $\mathcal{B} = \{u_1, u_2, \ldots, u_p\}$ is a basis of H if

$H = \operatorname{Col} A$ and $\operatorname{Nul} A = \{0\}$ where $A = \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix}$.

Theorem

If $\mathcal{B} = \{u_1, u_2, \ldots, u_p\}$ is a basis for $H \subseteq \mathbb{R}^n$ and

$\mathcal{C} = \{v_1, v_2, \ldots, v_q\}$ is another basis for H.

Then $p = q$ and $\dim H = p$ the common value of the number of vectors in any basis for H.

$$A = \left[u_1 \vdots u_2 \vdots \cdots \vdots u_p \right] \in \mathbb{R}^{n\times p} \qquad B = \left[v_1 \vdots v_2 \vdots \cdots \vdots v_q \right] \in \mathbb{R}^{n\times q}$$

Since $u_i \in H$ and the v_i's are a basis of H they span it and so there is a vector $c \in \mathbb{R}^q$ such that $u_i = Bc$ ← different c for each i

$$u_i = c_{1,i}v_1 + c_{2,i}v_2 + \cdots + c_{q,i}v_q$$

first c vector

$$\underbrace{\left[u_1 \vdots u_2 \vdots \cdots \vdots u_p \right]}_{A} = \underbrace{\left[v_1 \vdots v_2 \vdots \cdots \vdots v_q \right]}_{B} \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{q1} & c_{q2} & \cdots & c_{qp} \end{bmatrix}}_{C \in \mathbb{R}^{q\times p}}$$

Since $v_j \in H$ and the u_i's are a basis of H they span it and so there is a vector $d \in \mathbb{R}^p$ such that $v_j = Bd$ ← different d for each j

$$v_j = d_{1,j}u_1 + d_{2,j}u_2 + \cdots + d_{p,j}u_p$$

first c vector

$$\underbrace{\left[v_1 \vdots v_2 \vdots \cdots \vdots v_q \right]}_{B} = \underbrace{\left[u_1 \vdots u_2 \vdots \cdots \vdots u_p \right]}_{A} \underbrace{\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1q} \\ d_{21} & d_{22} & \cdots & d_{2q} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pq} \end{bmatrix}}_{D \in \mathbb{R}^{p\times q}}$$

Thus

substitute

$$A = BC \quad \text{and} \quad B = AD$$

substitute

$$A = (AD)C = A(DC) \quad \text{also} \quad B = (BC)D = B(CD)$$

Claim that $DC = I$ and also $CD = I$. $A = (AD)C = A(DC)$

$A = A(DC)$

$A - A(DC) = 0$

$AI - A(DC) = 0$

$A(I - DC) = 0$ means that $A(I - DC)x = 0$ for every x.

Something is wrong?

Suppose $DCx \neq x$ for some x

then $(I - DC)x \neq 0$

since $\text{Nul}\, A = \{0\}$

then $A(I - DC)x \neq 0$.

Thus $DCx = x$ for every x which means $DC = I$.

By symmetry also $CD = I$ and so C and D are square.

Thus $p = q$.