§2.9 #11

$$
\begin{array}{ccccc}
P & P & F & P & F
\end{array}
$$

11. $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix} \in \mathbb{R}^{4\times 5}$

we know there are free variables since there aren't enough rows to have a pivot in every column.

$$
\begin{array}{ccccc}
P & P & F & P & F
\end{array}
$$

$\sim \begin{bmatrix} \boxed{1} & 2 & -5 & 0 & -1 \\ 0 & \boxed{1} & 2 & 4 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ← echelon form of A

row equivalent to

pivots

a basis for Col A is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix} \right\}$

linearly independent and $\operatorname{col} A = \operatorname{span} \mathcal{B}$.

$\dim \operatorname{Col} A = 3 = \#$ of pivots,

$$\operatorname{Col} A = \{ Ax : x \in \mathbb{R}^5 \} = \operatorname{span} \mathcal{B} = \operatorname{Col} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 4 \\ -3 & -9 & -7 \\ 3 & 10 & 11 \end{bmatrix}$$

Now let's find

$$\operatorname{Nul} A = \{ x \in \mathbb{R}^5 : Ax = 0 \}$$

use Echelon form to solve $Ax = 0$. actually reduced echelon form

$$\begin{bmatrix} \boxed{1} & 2 & -5 & 0 & -1 \\ 0 & \boxed{1} & 2 & 4 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 - 2r_2$

$$\begin{bmatrix} 1 & 0 & -9 & -8 & -11 \\ 0 & \boxed{1} & 2 & 4 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 + 8r_3$

$r_2 \leftarrow r_2 - 4r_3$

reduced echelon form

$$
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & x_5
\end{array}
$$

$$\begin{bmatrix} 1 & 0 & -9 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 9x_3 + 5x_5 = 0$

$x_2 + 2x_3 - 3x_5 = 0$

$x_4 + 2x_5 = 0$

Solve for pivot variables in terms of the free

$$x_1 - 9x_3 + 5x_5 = 0 \qquad x_1 = 9x_3 - 5x_5$$
$$x_2 + 2x_3 - 3x_5 = 0 \qquad x_2 = -2x_3 + 3x_5$$
$$x_4 + 2x_5 = 0 \qquad x_4 = -2x_5$$

Solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 9x_3 - 5x_5 \\ -2x_3 + 3x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_5}_{\text{span of two vectors}}$$

$$A = \begin{array}{c} \begin{array}{ccccc} P & P & F & P & F \end{array} \\ \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix} \end{array}$$

these vectors are independent because the row corresponding to the free variable is different for each vector and equal 1 only for the vector for that variable.

$$\text{Nul } A = \text{span} \underbrace{\left\{ \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}}_{\text{basis for the Nul space}} = \text{Col} \underbrace{\begin{bmatrix} 9 & -5 \\ -2 & 3 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix}}_{N \text{ the nullspace matrix}}$$

dim Nul $A = 2$ = # of free variables

dim Col $A = 3$ = # of pivots,

dim Nul A + dim Col A = # of columns in A = 5

If $A \in \mathbb{R}^{m \times n}$ then dim Nul A + dim Col A = n.

§2.9 Theorem 14

The Rank Theorem

If a matrix A has n columns, then rank A + dim Nul $A = n$.

rank A = dim Col A by definition.

Summarize facts about inverses (in terms of these spaces).

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.
m. The columns of A form a basis of $\mathbb{R}^n$.
n. Col $A = \mathbb{R}^n$
o. rank $A = n$
p. dim Nul $A = 0$
q. Nul $A = \{\mathbf{0}\}$

3.1 Introduction to Determinants

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In general $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det \underset{n\times n}{A} &= a_{11} \det \underset{n-1\times n-1}{A_{11}} - a_{12} \det \underset{n-1\times n-1}{A_{12}} + \cdots + (-1)^{1+n} a_{1n} \det \underset{n-1\times n-1}{A_{1n}} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \underset{n-1\times n-1}{A_{1j}} \end{aligned}$$

Where $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the matrix given by crossing out the ith row and jth column of A.

Remark this is a recursive definition that defines the determinant of a $n\times n$ in terms of $(n-1)\times(n-1)$ determinants.