Recursive definition of determinant...

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\underbrace{\det A}_{n\times n} = \boxed{a_{11} \underbrace{\det A_{11}}_{n-1\times n-1} - a_{12} \underbrace{\det A_{12}}_{n-1\times n-1}} + \cdots + (-1)^{1+n} a_{1n} \underbrace{\det A_{1n}}_{n-1\times n-1}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \underbrace{\det A_{1j}}_{n-1\times n-1}$$

Where $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the matrix given by crossing out the ith row and jth column of A.

Example $A = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}$

Def: $\underbrace{\det A}_{3\times 3} = a_{11} \underbrace{\det A_{11}}_{2\times 2} - a_{12} \underbrace{\det A_{12}}_{2\times 2} + a_{13} \underbrace{\det A_{13}}_{2\times 2}$

$A_{11} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ $\qquad A_{12} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$

also $A_{13} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Therefore

$\det \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} - (-2) \det \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} + 1 \det \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$= 2\big((1)(-1) - (2)(3)\big) + 2\big((3)(-1) - (2)(1)\big) + \big((3)(3) - (1)(1)\big)$

$= (2)(-7) + 2(-5) + (8) = -14 - 10 + 8 = -16$

$\begin{array}{r} 24 \\ -8 \\ \hline 16 \end{array}$

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julia> A=[2 -2 1; 3 1 2; 1 3 -1]
3×3 Matrix{Int64}:
 2  -2   1
 3   1   2
 1   3  -1

julia> using LinearAlgebra

julia> det(A)
-16.0
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Using Julia to find determinant..

now use this

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

here i is constant

The cofactor expansion down the jth column is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

here j constant

$$\det A = \sum_{j=1}^{n} \underbrace{(-1)^{1+j}\, a_{1j} \det A_{1j}}_{C_{1j}} = \sum_{j=1}^{n} a_{1j} C_{1j}$$

same

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

i is constant

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

j is constant.

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

top row is here.

$$\det\begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} = 2\det\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} - (-2)\det\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} + 1\det\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$= \left(\det\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, -\det\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}, \det\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \right) \cdot (2, -2, 1)$$

Matrix vector mult.

$$= \begin{bmatrix} \det\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} & -\det\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} & \det\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Thus, $\det A$ is linear in the first row.

$$\det\begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} + \det\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} = \det\begin{bmatrix} 3 & 0 & 4 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

What do these formula mean.

$$\underbrace{\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}_{i \text{ const...}} \qquad \underbrace{\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}_{j \text{ const}}$$

$\det A$ is linear in the jth column.

$\det A$ is linear in the ith row.

Conclusion $\det A$ is multi-linear as a function of the rows of A
and also multi-linear as a function of the columns.

$$0 = \underbrace{\det\begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} + \det\begin{bmatrix} -2 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}}_{0} = \det\begin{bmatrix} 0 & 0 & 0 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

Since

$$\det\begin{bmatrix} -2 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} = (-2)\det\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} - (2)\det\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} + (-1)\det\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$= (-1)\left\{ 2\det\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} - (-2)\det\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} + 1\det\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \right\} = -\det\begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

Therefore

$$\det\begin{bmatrix}0 & 0 & 0\\ 3 & 1 & 2\\ 1 & 3 & -1\end{bmatrix} = 0\cdot\det\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix} - (0)\det\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix} + 0\det\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} = 0$$

The determinant of any matrix with a row of zeros is 0.

The determinant of any matrix with a column of zeros is 0.

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

Same except rows and columns are switched.

$$\det A^T = \sum_{j=1}^{n} (-1)^{i+j} a_{ji} \det A^T_{ij} = \det A$$

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} = ad - bc$$

$$\|$$

$$\det\begin{bmatrix}a & b\\ c & d\end{bmatrix}^T = \det\begin{bmatrix}a & c\\ b & d\end{bmatrix} = ad - cb = ad - bc$$

Summarize $\det A = \det A^T$, $\det A = 0$ if there is a zero column or row in A.

Use these observations (properties) about det A to come up with a better way of calculating.

Triangular matrix (usually come as echelon form of A)

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1\det\begin{bmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix} - 2\det\cancel{\begin{bmatrix} 0 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix}}$$

$$+ 3\det\cancel{\begin{bmatrix} 0 & 5 & 7 \\ 0 & 0 & 9 \\ 0 & 0 & 10 \end{bmatrix}} - 4\det\cancel{\begin{bmatrix} 0 & 5 & 6 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}}$$

Thus

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1\det\begin{bmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{bmatrix} = 1\cdot 5\cdot \det\begin{bmatrix} 8 & 9 \\ 0 & 10 \end{bmatrix}$$

$$= 1\cdot 5\cdot(8\cdot 10 - 9\cdot 0) = 1\cdot 5\cdot 8\cdot 10 = 400.$$

The determinant of a upper triangular matrix is the product of the entries on it's diagonal.