Find the LU factorization of A:

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

$r_2 \leftarrow r_2 - \frac{3}{2} r_1$

row col

$r_3 \leftarrow r_3 - \frac{1}{2} r_1$

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1/2 \\ 0 & 4 & -3/2 \end{bmatrix}$$

$r_3 \leftarrow r_3 - r_2$

$$U = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1/2 \\ 0 & 0 & -2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}$$

So $A = \cancel{UL}$ or LU? $A = LU$

$\det A = -16$ from last time

the same answer

$$\det U = \det \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1/2 \\ 0 & 0 & -2 \end{bmatrix} = 2 \cdot 4 \cdot (-2) = -16$$

$$\det L = \det \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} = 1$$

Haven't explained why but finding U first allows computing determinants of $A \in \mathbb{R}^{n \times n}$ in n^3 arithmetic operations rather than $n!$ by the def.

All row operations are invertible linear trans.

$$\det A = \underbrace{\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}_{i \text{ const...}} \qquad \det A = \underbrace{\sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}_{j \text{ const}}$$

How do row operations interact with the recursive formula for computing det A?

row swap. $\quad r_i \leftrightarrow r_k \quad\quad i \neq k$

Whats the matrix corresponding to the row swap?
Simply perform that row swap on the identity:
When $n = 3$ then

$$[r_1 \leftrightarrow r_3] = [r_1 \leftrightarrow r_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

What is $\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$?

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \qquad r_1 \leftrightarrow r_3$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$r_2 \leftarrow r_2 - 3r_1$
$r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -8 & 5 \\ 0 & -8 & 3 \end{bmatrix}$$

$r_3 \leftarrow r_3 - r_2$

$$U = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -8 & 5 \\ 0 & 0 & -2 \end{bmatrix} \qquad \det \begin{bmatrix} 1 & 3 & -1 \\ 0 & -8 & 5 \\ 0 & 0 & -2 \end{bmatrix} = 1(-8)(-2) = 16$$

almost the same as det A
except off by a minus sign

Whenever you swap rows it changes the determinant by a sign.

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}$$

Question why does a row swap change the sign...

Let $A \in \mathbb{R}^n$ and $r_i \leftrightarrow r_k$ swaps two rows of A

$B = [r_i \leftrightarrow r_k] A$

B is the same a A except with the i^{th} and k^{th} rows swapped.

$b_{kj} = a_{ij}$ for every j and $b_{ij} = a_{kj}$ for every j

$$\begin{array}{c} B \\ \begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} A \\ \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \end{array}$$

Simplifying assumption suppose the rows are adjacent ...

$$\begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix} \quad r_1 \leftrightarrow r_2$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -2 & 1 \\ 1 & 3 & -1 \end{bmatrix} \quad r_2 \leftrightarrow r_3$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & -2 & 1 \end{bmatrix} \quad r_1 \leftrightarrow r_2$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

try to make this

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

Same ...

Thus

$$[r_1 \leftrightarrow r_3] = \underbrace{[r_1 \leftrightarrow r_2][r_2 \leftrightarrow r_3][r_1 \leftrightarrow r_2]}_{\text{adjacent row swaps}}$$

If the rows are adjacent then $a_{ij} = c_{kj}$ and the minor matrices

$$A_{ij} = C_{kj} \quad \text{where} \quad C = [r_i \leftrightarrow r_k] A.$$

Example $n=3$ and $C = [r_1 \leftrightarrow r_2] A.$

cross out the 1st row and the 2nd column

cross out the second row and the 2nd column.

$$C_{12} = A_{22}$$

$$\begin{bmatrix} \cancel{3} & \cancel{1} & \cancel{2} \\ 2 & \cancel{-2} & 1 \\ 1 & \cancel{3} & -1 \end{bmatrix} . \qquad \begin{bmatrix} 2 & \cancel{-2} & 1 \\ \cancel{3} & \cancel{1} & \cancel{2} \\ 1 & \cancel{3} & -1 \end{bmatrix}$$

when the row swap $[r_i \leftrightarrow r_k]$ involves adjacent rows, then the entries $c_{kj} = a_{ij}$ and $c_{ij} = a_{kj}$ for all j. But also

The minor matrices

$$C_{kj} = A_{ij} \quad \text{and} \quad C_{ij} = A_{kj}$$

are also equal.