when the row swap $[r_i \leftrightarrow r_k]$ involves adjacent rows, then the entries $c_{kj} = a_{ij}$ and $c_{ij} = a_{kj}$ for all j. But also

The minor matrices

$$C_{kj} = A_{ij} \quad \text{and} \quad C_{ij} = A_{kj}$$

are also equal.

Recursive relation for determinant

$$\det A = \underbrace{\boxed{\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}}_{i \text{ const...}} \qquad \det A = \underbrace{\sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}}_{j \text{ const}}$$

Assume i and k are adjacent rows.. either $\boxed{k = i+1 \text{ or } k = i-1}$

$$C = [r_i \leftrightarrow r_k] A \qquad c_{kj} = a_{ij}, \quad c_{ij} = a_{kj} \qquad \text{also} \qquad C_{kj} = A_{ij}, \quad C_{ij} = A_{kj}$$

compare

$$\det C = \sum_{j=1}^{n} (-1)^{k+j} c_{kj} \det C_{kj} = \sum_{j=1}^{n} (-1)^{k+j} a_{ij} \det A_{ij}$$

(k const)

$$= \sum_{j=1}^{n} (-1)^{i+j-i+k} a_{ij} \det A_{ij} = (-1)^{-i+k} \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

$$= (-1)^{-i+k} \det A = -\det A$$

So even if non adjacent rows are swapped then the determinant still changes sign.

Determinant of row swap matrix

for any $i \neq k$ let $B = [r_i \leftrightarrow r_j] I$

$$\det[r_i \leftrightarrow r_k] = \det B = -\det I$$

Determinant of rescaling matrix.

$$B = [r_2 \leftarrow \alpha r_2] I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

I know determinant is linear in each row.

$$\det B = (-1)^3 b_{21} \det B_{21} + (-1)^4 b_{22} \det B_{22} + (-1)^5 b_{23} \det B_{23} + (-1)^6 b_{24} \det B_{24}$$

$$= -0 \det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 0 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \alpha$$

$$\det [r_i \leftarrow \alpha r_i] = \alpha$$

Determinant of an elimination step...

what about determinant of a matrix with two rows the same? suppose $r_i = r_k$

$$\det A = \det \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix} = -\det [r_i \leftrightarrow r_k] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix} = -\det \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix} = -\det A$$

Conclusion: if two rows are the same $\det A = 0$.

$$[r_2 \leftarrow r_2 - \alpha r_1]$$

$$\det [r_2 \leftarrow r_2 - \alpha r_1] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix} = \det \begin{bmatrix} r_1 \\ r_2 - \alpha r_1 \\ r_3 \\ \vdots \\ r_n \end{bmatrix}$$

linear in each row

$$= \det \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix} + \det \begin{bmatrix} r_1 \\ \alpha r_1 \\ r_3 \\ \vdots \\ r_n \end{bmatrix}$$

rescaling

$$= \det \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix} + \alpha \det \begin{bmatrix} r_1 \\ r_1 \\ r_3 \\ \vdots \\ r_n \end{bmatrix} = \det \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix}$$

two rows the same

Therefore $\det A = \det [r_i \leftarrow r_i - \alpha r_j] A$ for elimination step.

Summary:

$\det [r_i \leftrightarrow r_k] = -1$ $\qquad \det [r_i \leftrightarrow r_k] A = -\det A$

$\det [r_i \leftarrow \alpha r_i] = \alpha$ $\qquad \det [r_i \leftarrow \alpha r_i] A = \alpha \det A$

$\det [r_i \leftarrow r_i - \alpha r_j] = 1$ $\qquad \det [r_i \leftarrow r_i - \alpha r_j] A = 1 \det A$.

Also

$$\det [r_i \leftrightarrow r_k] A = \det [r_i \leftrightarrow r_k] \det A$$

$$\det [r_i \leftarrow \alpha r_i] A = \det [r_i \leftarrow \alpha r_i] \det A$$

$$\det [r_i \leftarrow r_i - \alpha r_j] A = \det [r_i \leftarrow r_i - \alpha r_j] \det A.$$

Since the product of determinants is the determinant of the product when the first matrix is an elementary matrix. Then the same holds for all matrices

$$\det BA = \det B \det A$$

or

$$\det AB = \det A \det B.$$

This is because any invertible matrix is the product of elementary row operations. Just take the row operations that were used to find the inverse and undo them all in reverse order.

Note is A is not invertible (and square) then the echelon form of A is missing pivots so there is a zero row. Since the determinant of the echelon form is equal to the original matrix A up to ± 1 due to the row swaps. Then $\det A = \pm \det U = \pm 0 = 0$.

Determinant of $I_i(x)$ where that the identity with the ith column replaced by the vector x

Example $n=3$.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad I_2(x) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$I_1(x) = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad I_3(x) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

What is $\det I_i(x) = x_i$