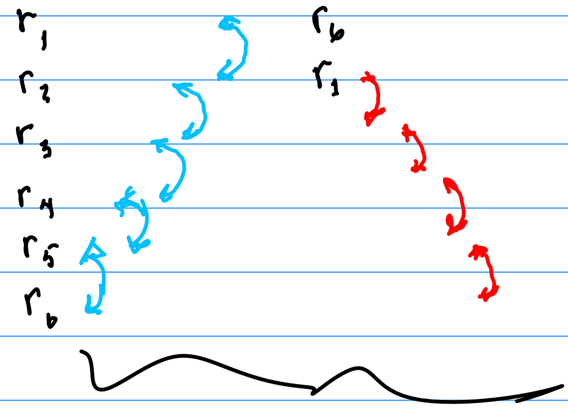
row swap



total # of adjacent row swaps is odd..

In the end the fact that $\det EA = \det E \det A$ for any matrix corresponding to a row operation. Implies

$$\det BA = \det B \det A \qquad \text{for any } n\times n \text{ matrices.}$$

Suppose $\det B \neq 0$. Then B is invertible and we find inverse using row operations

$$[\,B \mid I\,] \qquad E_1 = [r_2 \leftarrow r_2 - \alpha r_1]$$

$$E_2 = [r_3 \leftarrow r_3 - \beta r_1]$$

$$\vdots$$

$$E_{17} = [r_3 \leftrightarrow r_4]$$

$$\vdots$$

$$E_p$$

the

$$\downarrow$$

$$[\,I \mid B^{-1}\,]$$

$$E_p \cdots E_1 B = I$$

$$B = E_1^{-1} E_2^{-1} \cdots E_p^{-1} I$$

$$BA = E_1^{-1} E_2^{-1} \cdots E_p^{-1} A$$

$$\det BA = \det\left(E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1} A\right)$$

$$= \det E_1^{-1} \det\left(E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1} A\right)$$

$$\vdots$$

$$= \det E_1^{-1} \det E_2^{-1} \cdots \underbrace{\det E_{p-1}^{-1} \det E_p^{-1}} \det A$$

$$= \det E_1^{-1} \det E_2^{-1} \cdots \det\left(E_{p-1}^{-1} E_p^{-1}\right) \det A$$

$$\vdots$$

$$= \det\left(E_1^{-1} E_2^{-1} \cdots E_{p-1}^{-1} E_p^{-1}\right) \det A = \det B \det A.$$

Remark

$$1 = \det I = \det(BB^{-1}) = \det B \det B^{-1}$$

solve

$$\det B^{-1} = \frac{1}{\det B}.$$

Let $I_i(x)$ be the identity matrix with the ith column replaced by the vector x.

$$I = \left[e_1 \,\vdots\, e_2 \,\vdots\, \cdots \,\vdots\, e_n\right] = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{bmatrix}$$

$$I_i(x) = \left[e_1 \,\vdots\, \cdots \,\vdots\, e_{i-1} \,\vdots\, x \,\vdots\, e_{i+1} \,\vdots\, \cdots \,\vdots\, e_n\right] = \begin{bmatrix} 1 & & & x_1 & & & \\ & \ddots & & x_2 & & & \\ & & 1 & \vdots & & & \\ & & & x_i & & & \\ & & & x_{i+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & & & x_1 & & & \\ & \ddots & & x_2 & & & \\ & & & \vdots & & & \\ & & 1 & x_i & & & \\ & & & x_{i+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & & & x_1 & & & \\ & \ddots & & x_2 & & & \\ & & & \vdots & & & \\ & & 1 & x_i & & & \\ & & & 0 & 1 & & \\ & & & \vdots & & \ddots & \\ & & & 0 & & & 1 \end{bmatrix} + \det\begin{bmatrix} 1 & & & 0 & & & \\ & \ddots & & 0 & & & \\ & & & \vdots & & & \\ & & 1 & 0 & & & \\ & & & x_{i+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{bmatrix}$$

triangular matrix triangular matrix

$$= 1\cdot 1\cdots x_i \cdots 1 + 1\cdot 1\cdots 0\cdot 1\cdots 1 = x_i$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ x_{i+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_{i+1} \\ \vdots \\ x_n \end{bmatrix}$$

Conclusion $I_i(x) = x_i$

Usual question Solve $Ax = b$.

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \qquad A_i(b) = \begin{bmatrix} a_1 & \cdots & a_{i-1} & b & a_{i+1} & \cdots & a_n \end{bmatrix}$$

$$A\, I_i(x) = A\begin{bmatrix} e_1 & \cdots & e_{i-1} & x & e_{i+1} & \cdots & e_n \end{bmatrix} = \begin{bmatrix} Ae_1 & \cdots & Ae_{i-1} & Ax & Ae_{i+1} & \cdots & Ae_n \end{bmatrix}$$

Conclusion $A I_i(x) = A_i(b)$

$$\det(A I_i(x)) = \det A_i(b)$$

recall

$I_i(x) = x_i$

$\det A \det I_i(x) = \det A_i(b)$

$(\det A)\, x_i = \det A_i(b)$

Therefore $Ax = b$ means $x_i = \dfrac{\det A_i(b)}{\det A}$ for $i = 1, \dots, n$.

This is called Cramer's rule for solving $Ax = b$.

Chapter 3.3

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \dots, n \tag{1}$$

Inverse of A. Find the linear function A^{-1} so that $x = A^{-1}b$ means $Ax = b$.

$b = b_1 e_1 + b_2 e_2 + \dots + b_n e_n$

by linearity

$x = A^{-1}b = A^{-1}(b_1 e_1 + b_2 e_2 + \dots + b_n e_n)$

$= b_1 A^{-1} e_1 + b_2 A^{-1} e_2 + \dots + b_n A^{-1} e_n$

Need to find $A^{-1} e_j = v_j$ for each j

or solve $A v_j = e_j$ for each j

$$x = b_1 v_1 + b_2 v_2 + \dots + b_n v_n = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{A^{-1}} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

So what's left is to use Cramer's rule to find the v_j's.

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

$$v_{j,i} = \frac{\det A_i(e_j)}{\det A} \qquad \text{for } i = 1, 2, \ldots, n$$

Therefore.

$$A^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} \det A_1(e_1) & \det A_1(e_2) & \cdots & \det A_1(e_n) \\ \det A_2(e_1) & \det A_2(e_2) & & \\ \vdots & & \ddots & \\ \det A_n(e_1) & \cdots & \cdots & \det A_n(e_n) \end{bmatrix}$$

formula for A^{-1}