Find the reduced echelon form

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

$r_2 \leftarrow r_2 + r_1$

$r_3 \leftarrow r_3 - \frac{3}{2} r_1$

row

col.

$$\begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & -5 & -3 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$r_3 \leftarrow r_3 + r_2$

Practice LU factorization

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3/2 & -1 & 1 \end{bmatrix}$$

echelon form

$$U = \begin{bmatrix} \boxed{-2} & 4 & -2 & -4 \\ 0 & \boxed{-2} & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 + 2r_2$

$$\begin{bmatrix} \boxed{-2} & 0 & -12 & -10 \\ 0 & \boxed{-2} & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow -\frac{1}{2} r_1$

$r_2 \leftarrow -\frac{1}{2} r_2$

reduced echelon form

$$\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To solve $Ax = 0$ is the same as solving

$$\begin{array}{cccc} P & P & F & F \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 + 6x_3 + 5x_4 = 0$$

$$x_2 + 5/2\, x_3 + 3/2\, x_4 = 0$$

solve for pivot variables in terms of free variables

$x_1 = -6x_3 - 5x_4$

$x_2 = -5/2\, x_3 - 3/2\, x_4$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_3 - 5x_4 \\ -5/2\, x_3 - 3/2\, x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} x_4$$

all the solutions to $Ax=0$.

$$\text{Nul } A = \{x : Ax = 0\} = \left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} x_4 ;\ x_3, x_4 \in \mathbb{R} \right\}$$

this is a span

$$= \text{span}\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{Col} \begin{bmatrix} -6 & -5 \\ -5/2 & -3/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

each vector corresponds to a different free variable therefore the vectors are independent (because where the 1's and 0's are in relation to the free vbls).

a basis for Nul A is $\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Chapt 4.3

Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span } \mathcal{B}$$

$$A = \begin{array}{cccc} P & P & F & F \\ \end{array}$$

$$A=\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

pivot columns

$$\begin{array}{cccc} P & P & F & F \\ x_1 & x_2 & x_3 & x_4 \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form

How to find a basis for Col A?

Any dependency relation between the columns of the echelon form is also a dependency relation between the column of A

Every column of the (reduced) echelon form can be written as a linear combination of the pivot columns.

So the pivot columns of A form a basis for Col A

Basis of Col A is $\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$.

New subspace...

$$\boxed{\text{row } A = \text{Col } A^T} = \{A^T z : z \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

$$= \text{Span}\left\{ \begin{bmatrix} -2 \\ 4 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 2 \\ -3 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}^T = \begin{bmatrix} -2 & 2 & -3 \\ 4 & -6 & 8 \\ -2 & -3 & 2 \\ -4 & 1 & -3 \end{bmatrix}$$

One idea perform row operations on A^T to identify the basis of Col A^T.

$$A = \begin{array}{cccc} P & P & F & F \\ \end{array}$$
$$A=\left[\begin{array}{cccc} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{array}\right] \qquad \begin{array}{c|cccc} & P & P & F & F \\ & x_1 & x_2 & x_3 & x_4 \\ \hline & 1 & 0 & 6 & 5 \\ & 0 & 1 & 5/2 & 3/2 \\ & 0 & 0 & 0 & 0 \end{array}$$

pivot columns

reduced echelon form

Since row operations are invertible, then

$$\text{row}\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} = \text{row}\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

these rows form a lin indep. spanning set

$$\text{basis for row } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5/2 \\ 3/2 \end{bmatrix} \right\}$$

One WARNING note that ...

$$\text{Col}\begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \neq \text{col}\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Remark</u>: Just like a linear function can be represented by a matrix, so a subspace can be represented by the column space of a matrix.

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent, that is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$ $= \{Ax : x \in \mathbb{R}^n\}$

o. $\operatorname{rank} A = n$

p. $\dim \operatorname{Nul} A = 0$

q. $\operatorname{Nul} A = \{\mathbf{0}\}$ $= \{x : Ax = 0\}$

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

If $\mathcal{B}$ is a basis for V then

$V = \text{span}\,\mathcal{B}$ and $\mathcal{B}$ is linearly independent

Let

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

Equivalently

$V = \text{Col}\,B$ and $\text{Nul}\,B = \{0\}$.

$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ can be written $x = Bc$ where $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

If this representation of x were not unique there would be a vector $d \in \mathbb{R}^n$ such that also $x = Bd$.

$0 = x - x = Bc - Bd = B(c-d)$ so $c-d \in \text{Nul}\,B$

thus $c - d = 0$ and so $c = d$.

Therefore the c such that $x = Bc$ is unique...

Let $V \subseteq \mathbb{R}^m$ be a subspace, and

$\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis of V and

$\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ be another basis of V.

Since the dimension of the subspace is well defined, both these basis have the same number of vectors and

$\dim V = n$.

Talk about the connection between the two basis next time. That is, the change of basis matrix.