Finish finding $[B \leftarrow C]$ from before Spring break

$$[B \leftarrow C] = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}^{-1} = \frac{1}{9-8}\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

← Cramer's rule for a 2x2 matrix

cofactor matrix

$\frac{1}{\det A}$

5.1 Eigenvectors and Eigenvalues

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

Problem given $A \in \mathbb{R}^{n \times n}$ solve $Ax = \underbrace{\lambda x}_{\text{non-linear term.}}$ for x and λ

What is the point of eigenvectors and eigenvalues?

$$\underbrace{Ax}_{\text{matrix vector mult}} = \underbrace{\lambda x}_{\text{scalar vector mult.}}$$

The idea is to turn matrix multiplication into scalar mult.

In general there are n linearly ind eigenvectors for each matrix A and they form a basis for $\mathbb{R}^n$.

This basis turns matrix vector mult for any vector x into n scalar multiplications.

Suppose we solved $Ax = \lambda x$ for x and λ and got,

$$Av_1 = \lambda_1 v_1 \qquad \text{and} \qquad Av_2 = \lambda_2 v_2$$

Suppose $x \in \operatorname{span}\{v_1, v_2\}$.

Then $x = c_1 v_1 + c_2 v_2$.

$$Ax = A(c_1 v_1 + c_2 v_2) = Ac_1 v_1 + Ac_2 v_2$$

$$= c_1 A v_1 + c_2 A v_2 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$$

two scalar multiplications instead of the matrix mult

How to solve $Ax = \lambda x$?

A computer would solve this problem by guessing a change of basis matrix so that matrix mult by A turns into n scalar multiplications. The guesses would be iteratively improved until the error is good enough...

We will use the theory of determinants.

$Ax = \lambda x$ trying to find λ and $x \neq 0$.

$Ax - \lambda I x = 0$ (factor x)

$(A - \lambda I)x = 0$ thus $x \in \operatorname{Nul}(A - \lambda I)$

definition of nullspace.

Note that $\text{Nul}(A-\lambda I)$ contains non-zero vectors only when $A-\lambda I$ has free variables.

Since A is square then $A-\lambda I$ is square and so free variables means there is not a pivot in every row.

Theory of determinants.

Echelon form of $A-\lambda I$ has some zero rows at the bottom.

$$\det(A-\lambda I)=0 \quad \text{when} \quad \text{Nul}(A-\lambda I) \text{ is non-trivial.}$$

Idea: solve $\det(A-\lambda I)=0$ for λ.

$n=2$

$$\det A = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Thus,

$$\det(A-\lambda I) = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= \underbrace{(a-\lambda)(d-\lambda) - bc}_{\text{quadratic polynomial in } \lambda} = 0$$

quadratic polynomial in λ. We can solve for λ using the quadratic formula...

$n=3$

$$\det A = \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

note: this pattern falls apart for 4x4 matrix and larger

$$\det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{pmatrix}$$

$$= (a-\lambda)(e-\lambda)(i-\lambda) + bfg + cdh - (a-\lambda)fh - bd(i-\lambda) - c(e-\lambda)g$$

this is a third degree polynomial in λ.

in general $\det(A - \lambda I)$ is a n^{th} degree polynomial in λ when $A \in \mathbb{R}^{n \times n}$.

Remark about triangular matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \det A = 1 \cdot 4 \cdot 6 = 24$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda)(6-\lambda) = 0$$

therefore $\lambda = 1$ or 4 or 6.

note the diagonal entries are the eigenvalues...

So it's easy to convert matrix vector multiplication by a triangular matrix to scalar multiplications...