$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \det A = 1 \cdot 4 \cdot 6 = 24$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda)(6-\lambda) = 0$$

therefore $\lambda = 1$ or $4$ or $6$.

Now find $x \neq 0$ in $\mathrm{Nul}(A - \lambda I)$ for each $\lambda$

$\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

Find reduced row echelon form to solve $(A - \lambda I)x = 0$.

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \qquad r_2 \leftarrow r_2 - \frac{3}{2} r_1$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 1/2 \\ 0 & 0 & 5 \end{bmatrix} \qquad r_3 \leftarrow r_3 + 10 r_2$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_2 \leftarrow 2 r_2 \quad \text{(rescale)}$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad r_1 \leftarrow r_1 - 3 r_2$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow \frac{1}{2} r_1$

reduced echelon form

read off the solution

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \text{free}$

$x_2 = 0$

$x_3 = 0$

$$x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1$$

$$\mathrm{Nul}(A - 1I) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Eigenspace of the eigenvalue $\lambda = 1$.

For example $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 1$

same vector ... eigenvector

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

eigenvalue

$\lambda = 4$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

Now find the nullspace...

$$\begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$r_3 \leftarrow r_3 - \frac{2}{5} r_2$

$r_1 \leftarrow r_1 - \frac{3}{5} r_2$

$$\begin{bmatrix} -3 & 2 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow -\frac{1}{3} r_1$

$r_2 \leftarrow \frac{1}{5} r_2$

reduced echelon form

$$\begin{array}{ccc} P & F & P \end{array}$$
$$\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solve for pivots in terms of free variables

$$x_1 - \frac{2}{3}x_2 = 0$$
$$x_3 = 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} x_2$$

$$\text{Nul}(A-4I) = \text{span}\left\{ \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$$

both eigenvectors for $\lambda = 4$

might want a unit eigenvector. (rescale so norm is 1).

$\lambda = 6$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow r_1 + r_2$

$$\begin{bmatrix} -5 & 0 & 8 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$r_1 \leftarrow -\frac{1}{5}r_1$

$r_2 \leftarrow -\frac{1}{2}r_2$

reduced echelon form

$$\begin{bmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{8}{5}x_3$$
$$x_2 = \frac{5}{2}x_3$$
$$x_3 = x_3$$

$$\text{Nul}(A-\lambda I) = \text{span}\left\{ \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix} \right\}$$
$$= \text{span}\left\{ \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix} \right\}$$

an eigenvector

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 16+50+30 \\ 100+50 \\ 60 \end{bmatrix} = \begin{bmatrix} 96 \\ 150 \\ 60 \end{bmatrix} = 6\begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}$$

eigenvalue

$$\left(\lambda_1 = 1, v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \text{ and } \left(\lambda_2 = 4, v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\right) \text{ and } \left(\lambda_3 = 6, v_3 = \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}\right)$$

Chapter 5.1 Theorem 2

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix $A$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

read proof for next time

$$\mathcal{B} = \{e_1, e_2, e_3\} \qquad \mathcal{C} = \{v_1, v_2, v_3\}.$$

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad \mathcal{C} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}\right\}$$

Change of basis matrix

$$[\mathcal{B} \leftarrow \mathcal{C}] = \left[ [v_1]_{\mathcal{B}} \,\vdots\, [v_2]_{\mathcal{B}} \,\vdots\, [v_3]_{\mathcal{B}} \right] = \left[ v_1 \,\vdots\, v_2 \,\vdots\, v_3 \right] = \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}$$

$$[\mathcal{C} \leftarrow \mathcal{B}] = [\mathcal{B} \leftarrow \mathcal{C}]^{-1} = \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}^{-1}$$

Idea write the linear function represented by $A$ in the basis $\mathcal{C}$.

$Ax$

$$x = [x]_\beta = [\beta \leftarrow e][x]_e$$

$$[Ax]_e = [e \leftarrow \beta]\,[Ax]_\beta = [e \leftarrow \beta]\,Ax$$

$$= \underbrace{[e \leftarrow \beta]\,A \cdot [\beta \leftarrow e]}\,[x]_e$$

... this is the representation of A in the e basis.