$$A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

$$\lambda_1 = 2 - 3i \qquad v_1 = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix} \qquad \qquad \lambda_2 = 2 + 3i \qquad v_2 = \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$$

$$\alpha = 2, \quad \beta = -3, \qquad u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$A \in \mathbb{R}^{2 \times 2}$$

$$\lambda_1 = \alpha + i\beta \qquad v_1 = u + iv \qquad \qquad \lambda_2 = \alpha - i\beta \qquad v_2 = u - iv$$

Change basis from $\mathcal{B} = \{e_1, e_2\}$ to $\mathcal{C} = \{v_1, v_2\}$

$$[\mathcal{B} \leftarrow \mathcal{C}] = \Big[[v_1]_{\mathcal{B}} \,\vdots\, [v_2]_{\mathcal{B}} \Big] = \big[v_1 \,\vdots\, v_2 \big] = \begin{bmatrix} 1+3i & 1-3i \\ 2 & 2 \end{bmatrix}$$

$$[\mathcal{C} \leftarrow \mathcal{B}] \, A \, [\mathcal{B} \leftarrow \mathcal{C}] = D = \begin{bmatrix} 2-3i & 0 \\ 0 & 2+3i \end{bmatrix}$$

The fact that factoring a real matrix could lead to complex numbers is not surprising, because complex numbers came from factoring things in the first place

$$x^2 + 4 = (x - 2i)(x + 2i)$$

This $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is also basis....

$$\mathcal{D} = \{u, v\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$$

Idea write A with respect to the basis $\mathcal{D}$.

Find $[\mathcal{D} \leftarrow \mathcal{B}] A [\mathcal{B} \leftarrow \mathcal{D}]$

Cramer's rule for 2x2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & c \end{bmatrix}$$

$$[\mathcal{B} \leftarrow \mathcal{D}] = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$[\mathcal{D} \leftarrow \mathcal{B}] = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}^{-1} = \frac{1}{-6} \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

$$[\mathcal{D} \leftarrow \mathcal{B}] A [\mathcal{B} \leftarrow \mathcal{D}] = \frac{1}{-6} \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$= \frac{-1}{6} \begin{bmatrix} 6 & -9 \\ -4 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = \frac{-1}{6} \begin{bmatrix} -12 & 18 \\ -18 & -12 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

not diagonal because we didn't use an eigenbasis, but there is still a pattern.

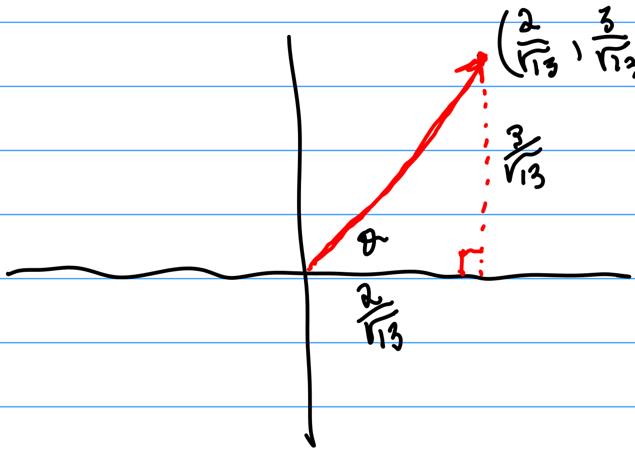
Polar coordinates

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

$$\underbrace{2^2 + 3^2}_{\substack{4+9 \\ 13}} = r^2(\cos^2\theta + \sin^2\theta) = r^2 \qquad r = \sqrt{13}$$

$$\begin{bmatrix}2\\3\end{bmatrix} = r\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \qquad \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2/\sqrt{13}\\ 3/\sqrt{13}\end{bmatrix}$$

this is in the first quadrant



$\tan\theta = \frac{3}{2}$

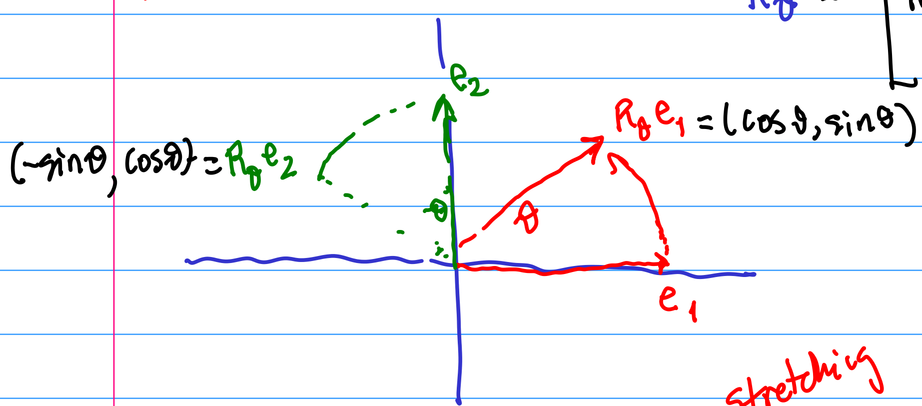
or $\theta = \arctan 3/2$

R_θ rotation.

$$\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix} = \sqrt{13}\begin{bmatrix}2/\sqrt{13} & -3/\sqrt{13}\\ 3/\sqrt{13} & 2/\sqrt{13}\end{bmatrix} = \sqrt{13}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

recall

$$R_\theta = \left[\, R_\theta e_1 \;\vdots\; R_\theta e_2 \,\right] = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$



stretching

rotation.

$$[D \leftarrow B]\, A\, [B \leftarrow D] = r \; R_\theta$$

physical interpretation of complex eigenvalues is a rotation followed by stretching (or contracting).

$A \in \mathbb{R}^{2\times 2}$

$\lambda_1 = \alpha + i\beta \quad v_1 = u + iv \qquad\qquad \lambda_2 = \alpha - i\beta \quad v_2 = u - iv$

$S = \left[u \;\vdots\; v \right] \qquad\qquad S^{-1}AS$ = stretch and rotate.

$Au + iAv = Av_1 = \lambda_1 v_1 = (\alpha + i\beta)(u + iv) = \alpha u - \beta v + i(\beta u + \alpha v)$

Therefore $Au = \alpha u - \beta v$ and $Av = \beta u + \alpha v$

$$AS = A\left[u \;\vdots\; v \right] = \left[Au \;\vdots\; Av \right] = \left[\alpha u - \beta v \;\vdots\; \beta u + \alpha v \right]$$

$$= \left[u \;\vdots\; v \right] \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

Therefore

$$S^{-1}AS = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \sqrt{\alpha^2 + \beta^2} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Foreshadowing of chapter 7.

If $A \in \mathbb{R}^{n\times n}$ with $A = A^T$ then the eigenvalues of A are real and there is an eigenbasis consisting of ortho-normal eigenvectors... Thus $Av_i = \lambda_i v_i$ for $i = 1, 2, \ldots, n$

and $v_i \cdot v_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$