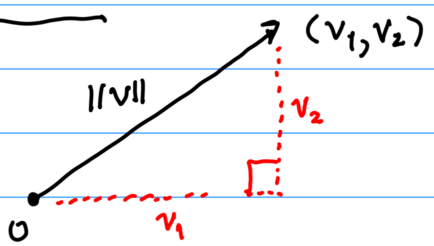
Chapter 6.

Review <u>dot product</u>

$u, v \in \mathbb{R}^n$ then $u \cdot v = \sum_{i=1}^{n} u_i v_i$

<u>Vector norm</u>.



Pythagorean theorem

$\|v\|^2 = v_1^2 + v_2^2$

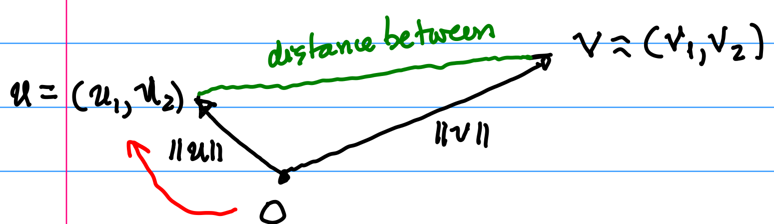
then

$\|v\| = \sqrt{v_1^2 + v_2^2}$

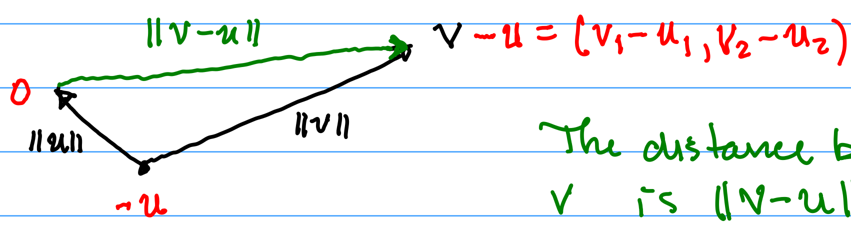
So if $v \in \mathbb{R}^n$ then $\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{v \cdot v}$

Remark if $v \in \mathbb{C}^n$ then $\|v\| = \sqrt{\sum_{i=1}^{m} |v_i|^2} = \sqrt{\sum_{i=1}^{n} \bar{v}_i v_i} = \sqrt{\bar{v} \cdot v}$.

Distance between two vectors



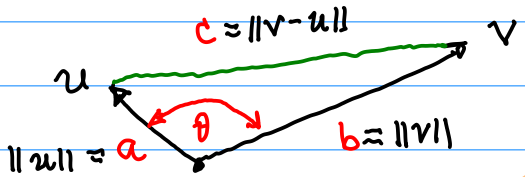
Translate the origin and then use Pythagorean theorem



The distance between u and v is $\|v - u\|$.

Remark $\|u - v\|$ is the same thing since length doesn't depend on the direction the vector is pointing.

Law of Cosines..



$$a^2 + b^2 = c^2 + 2ab\cos\theta$$

$$\|u\|^2 + \|v\|^2 = \|v-u\|^2 + 2\|u\|\|v\|\cos\theta$$

$$\|v-u\|^2 = (v-u)\cdot(v-u) = v\cdot v + u\cdot u - \underbrace{v\cdot u - u\cdot v}_{\text{same}}$$

$$= \|v\|^2 + \|u\|^2 - 2v\cdot u$$

or

$$= \|v\|^2 + \|u\|^2 - 2u\cdot v$$

Therefore $u\cdot v = \|u\|\|v\|\cos\theta$ where θ is the angle between u and v. (Geometric interpretation of dot product)

Remark: If u is perpendicular (orthogonal) to v then $\boxed{u\cdot v = 0}$.

Remark: If u and v are unit vectors then $\|u\| = 1$ and $\|v\| = 1$ so $u\cdot v = \cos\theta$.

Some algebra:

If $A, B \in \mathbb{R}^{n\times n}$ and $BA = I$ then $AB = I$

(True/False?) If true explain why. If false provide an example to show it is false.

TRUE!

Explanation 1

Since $BA = I$ then the range of B must be all of $\mathbb{R}^n$.

Thus $\operatorname{Col} B = \mathbb{R}^n$ ← from number of rows of B

Since $\dim \operatorname{Col} B + \dim \operatorname{Nul} B = n$ ← from the number of columns of B

then $\dim \operatorname{Nul} B = 0$

Thus $\operatorname{Nul} B = \{0\}$.

Now $BA = I$ multiply on the left by B

$$BAB = B$$

so $BABx = Bx$ for all $x \in \mathbb{R}^n$

$$BABx - Bx = 0$$

$B(ABx - x) = 0$ thus $ABx - x \in \operatorname{Nul} B$

Since $\operatorname{Nul} B = \{0\}$ then $ABx - x = 0$ so $ABx = x$

for all x this means

$AB = I$ $\boxed{\checkmark}$.

Explanation 2

Since $BA = I$ then the nullspace of A must be $\operatorname{Nul} A = \{0\}$.

Thus $\operatorname{Col} A = \mathbb{R}^n$ (Same arg. as before except for A.)

Since $\operatorname{Col} A = \mathbb{R}^n$ then for any $y \in \mathbb{R}^n$ there is x such that $y = Ax$.

$$BA = I$$

$$BAx = x$$

$$By = x$$

$ABy = Ax = y$ for all y so $AB = I$.

Let $\{u_1, \dots, u_n\} \in \mathbb{R}^m$ be a set of orthonormal vectors. What does that mean?

$$u_i \cdot u_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Remark: $\|u_i\| = \sqrt{u_i \cdot u_i} = 1$ so u_i are unit vectors.

Remark: The standard basis e_i form a set of orthonormal vectors.

Put vectors in a matrix

$$U = \begin{bmatrix} u_1 \vdots u_2 \vdots \cdots \vdots u_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Now

$$U^T U = \begin{bmatrix} \cdots u_1^T \cdots \\ \cdots u_2^T \cdots \\ \vdots \\ \cdots u_n^T \cdots \end{bmatrix} \begin{bmatrix} u_1 \vdots u_2 \vdots \cdots \vdots u_n \end{bmatrix} = \begin{bmatrix} u_1^T u_1 & u_1^T u_2 & \cdots & u_1^T u_n \\ u_2^T u_1 & u_2^T u_2 & \cdots & u_2^T u_n \\ \vdots & & & \\ u_n^T u_1 & u_n^T u_2 & \cdots & u_n^T u_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \cdots & u_1 \cdot u_n \\ u_2 \cdot u_1 & u_2 \cdot u_2 & \cdots & u_2 \cdot u_n \\ \vdots & & & \\ u_n \cdot u_1 & u_n \cdot u_2 & \cdots & u_n \cdot u_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

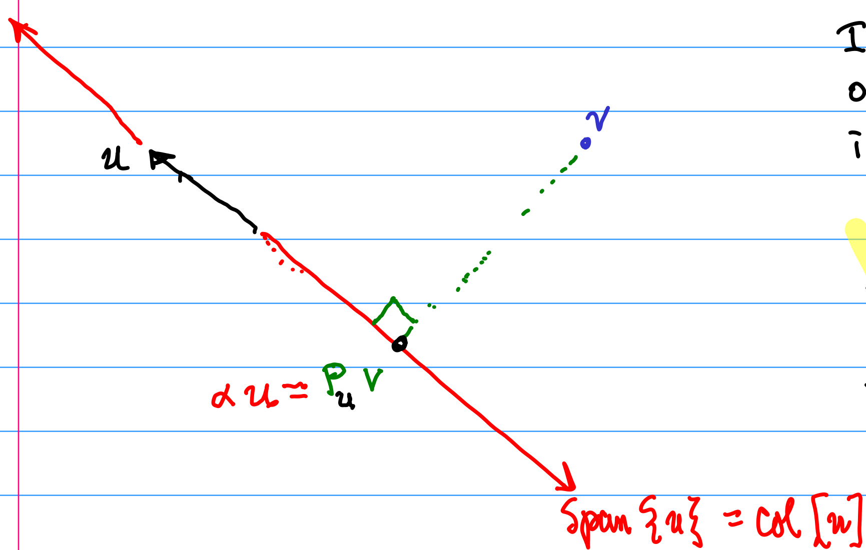
Remark if $m=n$ then U^T and U are square so $U^{-1} = U^T$ in this case we say U is an orthogonal matrix.

Result! A square matrix with orthonormal columns is an orthogonal matrix.

If U is not square then $U^TU = I$ but UU^T is not. What is UU^T? It is the orthogonal projection onto the subspace spanned by the vectors u_i. I.e. the projection onto

$$\operatorname{Col} U = \operatorname{span}\{u_1, u_2, \dots, u_n\}.$$

Consider the one-dimensional case first.



I want $v - P_u v$ to be orthogonal to any point in $\operatorname{span}\{u\}$.

$$\beta u \cdot (v - P_u v) = 0$$

$$u \cdot (v - \alpha u) = 0$$

$$u \cdot v - u \cdot \alpha u = 0$$

$$u \cdot v = \alpha\, u \cdot u$$

$$\alpha = \frac{u \cdot v}{u \cdot u}$$

Thus $P_u v = \alpha u = \dfrac{u \cdot v}{u \cdot u} u = \dfrac{u^T v}{u^T u} u = u \dfrac{u^T v}{u^T u}$

$$= \underbrace{\frac{u u^T}{u^T u}}\, v$$

Thus $P_u = \dfrac{u u^T}{u^T u}$