Summary:

- Given a basis $\{u_1, u_2, \dots, u_p\} \subseteq \mathbb{R}^n$ for a subspace $W \subseteq \mathbb{R}^n$ the orthogonal projection $P : \mathbb{R}^n \to W$ is given

$$P = U(U^T U)^{-1} U^T$$

where $U = \left[u_1 \vdots u_2 \vdots \cdots \vdots u_p \right]$.

- If basis is orthogonal $u_i \cdot u_j = \begin{cases} \|u_i\|^2 > 0 & \text{if } i=j \\ 0 & \text{if } i=j \end{cases}$

Then $P = P_{u_1} + P_{u_2} + \cdots + P_{u_p}$ where P_{u_i} is the projection onto the span $\{u_i\}$.

$$P_{u_i} v = \frac{u_i \cdot v}{\|u_i\|^2} u_i = \frac{u_i u_i^T}{\|u_i\|^2} v \qquad P_{u_i} = \frac{u_i u_i^T}{\|u_i\|^2}$$

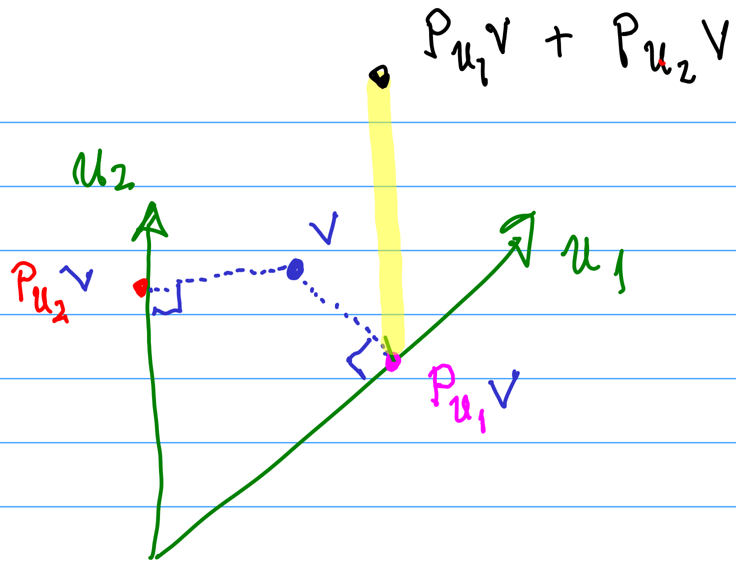
- If basis is orthonormal $u_i \cdot u_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i=j \end{cases}$

Then $U^T U = I$ and $P = U U^T$

and $P_{u_i} v = (u_i \cdot v) u_i = u_i u_i^T v \qquad P_{u_i} = u_i u_i^T$.

Warning if $\{u_1, u_2, \dots u_p\}$ is not orthogonal then

$$P \neq P_{u_1} + P_{u_2} + \cdots + P_{u_p}$$



$P \cdot v$ the the projection onto the page, then

$Pv \neq P_{u_1}v + P_{u_2}v$

Idea given any basis $\{u_1, u_2, \ldots, u_p\}$ find an orthogonal basis $\{v_1, v_2, \ldots, v_p\}$ for the same subspace and a ortho-normal basis $\{q_1, q_2, \ldots, q_n\}$ for the same subspace.

Last step $q_1 = \frac{v_1}{\|v_1\|}$, $q_2 = \frac{v_2}{\|v_2\|}$, $\ldots$, $q_p = \frac{v_p}{\|v_p\|}$

First step... find the v_i ... Gram-schmidt algorithm.

Gaussian elimination

row operation

two triangular matrices

row operation for every equation and every variable.

about n^2 rows operations.

$n \cdot n^2$ total arithmetic operation

Gram-Schmidt

column operations

one orthogonal matrix and one triangular.

Make a projection for each column

projections are matrix-vector multiplication n^2 arithmetic operation.

$n \cdot n^2$ total arithmetic operations.

also n square roots for the norms...

Gram-Schmidt orthogonalization algorithm..

Given $\{u_1, \dots, u_p\}$ find $\{v_1, \dots, v_p\}$

$v_1 = u_1$ $\qquad$ $\operatorname{span}\{v_1\} = \operatorname{span}\{u_1\}$

nothing extra here.

$v_2 = u_2 - P_{v_1} u_2$

$$\operatorname{span}\{v_1, v_2\} = \operatorname{span}\{u_1, u_2 - \underset{\text{proj out } v_1}{P_{v_1} u_2}\} = \operatorname{span}\left\{u_1, u_2 - \frac{v_1 \cdot u_1}{v_1 \cdot v_1} v_1\right\}$$

$$= \operatorname{span}\{u_1, u_2\}$$

$$v_3 = u_3 - \underbrace{P_{v_1} u_3 - P_{v_2} u_3}$$

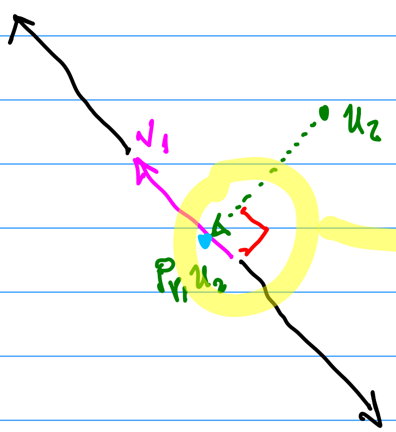
this is the projection onto $\operatorname{span}\{v_1, v_2\}$ because v_1 and v_2 are orthogonal.

$$v_1 \cdot v_2 = v_1 \cdot (u_2 - P_{v_1} u_2) = 0$$

in span of v_1

closest point in v_1 to u_2

since $P_{v_1} u_2$ is the orthogonal projection



Span$\{v_1\}$

$\vdots$

$$v_p = u_p - \underbrace{P_{v_1} u_p - P_{v_2} u_p - \cdots - P_{v_{p-1}} u_p}$$

projection onto the span $\{v_1, \dots, v_{p-1}\}$.

thus $v_p \cdot v_i = 0$ for any $i = 1, 2, \dots, p-1$

What we just wrote out

Theorem 11 in section 6.4

The Gram–Schmidt Process

Given a basis $\{u_1, \ldots, u_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = u_1$$

$$\mathbf{v}_2 = u_2 - \frac{u_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

$$\mathbf{v}_3 = u_3 - \frac{u_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{u_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$\vdots$$

$$\mathbf{v}_p = \mathbf{x}_p - \frac{u_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{u_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{u_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \quad \text{for } 1 \le k \le p \tag{1}$$

Modify algorithm to create $q_i = \frac{v_i}{\|v_i\|}$ ...

$v_1 = u_1$ $\qquad q_1 = \frac{v_1}{\|v_1\|}$

$v_2 = u_2 - (q_1 \cdot u_2)\, q_1$ $\qquad q_2 = \frac{v_2}{\|v_2\|}$

$v_3 = u_3 - (q_1 \cdot u_3)q_1 - (q_2 \cdot u_3)q_2$ $\qquad q_3 = \frac{v_3}{\|v_3\|}$

$\vdots$

$v_p = u_p - (q_1 \cdot u_p)q_1 - (q_2 \cdot u_p)q_2 - \cdots - (q_{p-1} \cdot u_p)\, q_{p-1}$

$q_p = \frac{v_p}{\|v_p\|}$

Then $\{q_1, q_2, \ldots, q_p\}$ is an orthonormal basis...

Write u_i's in terms of q_i's to find a matrix factorization given by this algorithm...

Assumption I know all the dot products, the norms and the q_i's ... Now solve for the u_i's

$v_1 = u_1$ $\qquad q_1 = \dfrac{v_1}{\|v_1\|}$

$u_1 = v_1 = \|v_1\| q_1$ $\qquad v_1 = \|v_1\| q_1$

$v_2 = u_2 - (q_1 \cdot u_2) q_1$ $\qquad q_2 = \dfrac{v_2}{\|v_2\|}$

$u_2 = (q_1 \cdot u_2) q_1 + \|v_2\| q_2$

$v_3 = u_3 - (q_1 \cdot u_3) q_1 - (q_2 \cdot u_3) q_2$ $\qquad q_3 = \dfrac{v_3}{\|v_3\|}$

$u_3 = (q_1 \cdot u_3) q_1 + (q_2 \cdot u_3) q_2 + \|v_3\| q_3$

$\vdots$

$v_p = u_p - (q_1 \cdot u_p) q_1 - (q_2 \cdot u_p) q_2 - \cdots - (q_{p-1} \cdot u_p) q_{p-1}$

$q_p = \dfrac{v_p}{\|v_p\|}$

$u_p = (q_1 \cdot u_p) q_1 + (q_2 \cdot u_p) q_2 + \cdots + (q_{p-1} \cdot u_p) q_{p-1} + \|v_p\| q_p$

$$\underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ u_1 & u_2 & \cdots & u_p \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} \vdots & \vdots & & \vdots \\ q_1 & q_2 & \cdots & q_p \\ \vdots & \vdots & & \vdots \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \|v_1\| & q_1 \cdot u_2 & & q_1 \cdot u_p \\ 0 & \|v_2\| & & q_2 \cdot u_p \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & & \|v_p\| \end{bmatrix}}_{R}$$

$U = Q\,R$

orthogonal