Example 3.2. A suburban community intends to replace its old fire station with a new facility. The old station was located at the historical city center. City planners intend to locate the new facility more scientifically. A statistical analysis of response–time data yielded an estimate of $3.2 + 1.7r^{0.91}$ minutes required to respond to a call r miles away from the station. (The derivation of this formula is the subject of Exercises 17 and 18 in Chapter 8.) Estimates of the frequency of calls from different areas of the city were obtained from the fire chief. They are presented in Figure 3.7. Each block represents one square mile, and the numbers inside each block represent the number of emergency calls per year for that block. Find the best location for the new facility.

$$
\begin{aligned}
z = 3.2 + 1.7\,[&6\sqrt{(x-1)^2+(y-5)^2}^{\,0.91} \\
&+8\sqrt{(x-3)^2+(y-5)^2}^{\,0.91} + 8\sqrt{(x-5)^2+(y-5)^2}^{\,0.91} \\
&+21\sqrt{(x-1)^2+(y-3)^2}^{\,0.91} + 6\sqrt{(x-3)^2+(y-3)^2}^{\,0.91} \\
&+3\sqrt{(x-5)^2+(y-3)^2}^{\,0.91} + 18\sqrt{(x-1)^2+(y-1)^2}^{\,0.91} \\
&+8\sqrt{(x-3)^2+(y-1)^2}^{\,0.91} + 6\sqrt{(x-5)^2+(y-1)^2}^{\,0.91}]/84.
\end{aligned}
$$

y

(i=1) 5

(i=2) 3

(i=3) 1

3	0	1	4	2	1
2	1	1	2	3	2
5	3	3	0	1	2
8	5	2	1	0	0
10	6	3	1	3	1
0	2	3	1	1	1

1 j=1   3 j=2   5 j=3

```
julia> A=[6 8 8; 21 6 3; 18 8 6]
3×3 Matrix{Int64}:
  6  8  8
 21  6  3
 18  8  6

julia> using Symbolics

julia> @variables x,y
2-element Vector{Num}:
 x
 y
```

Map array indices ...

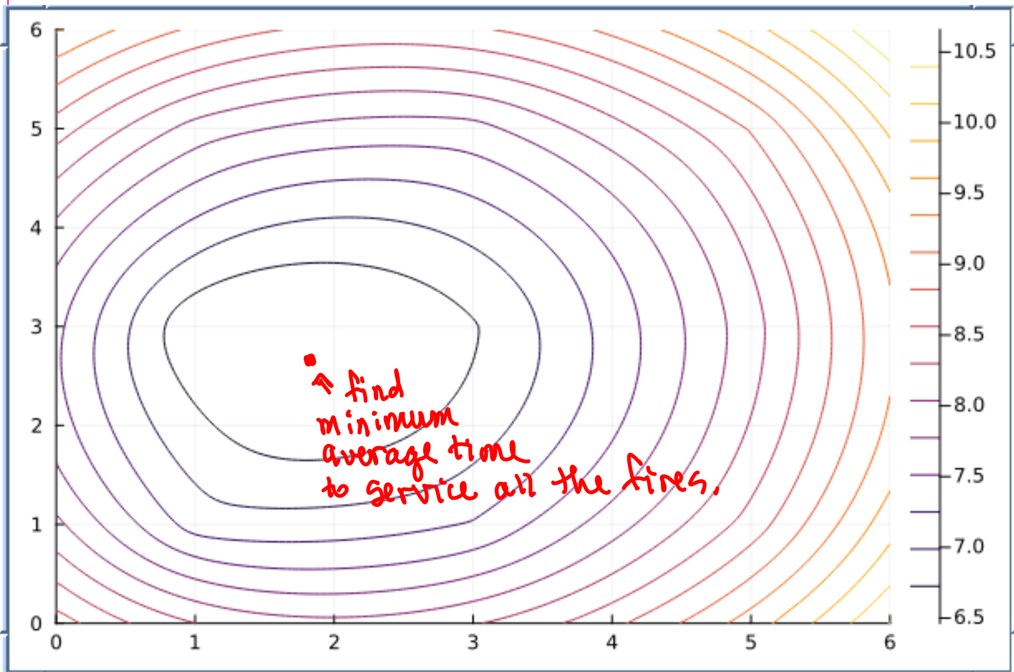
```
julia> toy(i)=7-2*i
toy (generic function with 1 method)

julia> tox(j)=2*j-1
tox (generic function with 1 method)
```

```
julia> R=[sqrt((x-tox(j))^2+(y-toy(i))^2) for i=1:3,j=1:3]
3×3 Matrix{Num}:
 sqrt((-1 + x)^2 + (-5 + y)^2)  …  sqrt((-5 + x)^2 + (-5 + y)^2)
 sqrt((-1 + x)^2 + (-3 + y)^2)     sqrt((-5 + x)^2 + (-3 + y)^2)
 sqrt((-1 + x)^2 + (-1 + y)^2)     sqrt((-5 + x)^2 + (-1 + y)^2)

julia> z=3.2+1.7*sum(A.*R.^0.91)/84
3.2 + 0.02023809523809524(18(sqrt((-1 + x)^2 + (-1 + y)^2)^0.91)
 + 8(sqrt((-3 + x)^2 + (-1 + y)^2)^0.91) + 8(sqrt((-5 + x)^2 + (
-5 + y)^2)^0.91) + 6(sqrt((-3 + x)^2 + (-3 + y)^2)^0.91) + 3(sqr
t((-5 + x)^2 + (-3 + y)^2)^0.91) + 6(sqrt((-1 + x)^2 + (-5 + y)^
2)^0.91) + 21(sqrt((-1 + x)^2 + (-3 + y)^2)^0.91) + 6(sqrt((-5 +
 x)^2 + (-1 + y)^2)^0.91) + 8(sqrt((-3 + x)^2 + (-5 + y)^2)^0.91
))
```

Non-linear objective function...

```
julia> fs="fz(x,y)="*string(z)
"fz(x,y)=3.2 + 0.02023809523809524(18(sqrt((-1 + x)^2 + (-1 + y)
^2)^0.91) + 8(sqrt((-3 + x)^2 + (-1 + y)^2)^0.91) + 8(sqrt((-5 +
 x)^2 + (-5 + y)^2)^0.91) + 6(sqrt((-3 + x)^2 + (-3 + y)^2)^0.91
) + 3(sqrt((-5 + x)^2 + (-3 + y)^2)^0.91) + 6(sqrt((-1 + x)^2 +
(-5 + y)^2)^0.91) + 21(sqrt((-1 + x)^2 + (-3 + y)^2)^0.91) + 6(s
qrt((-5 + x)^2 + (-1 + y)^2)^0.91) + 8(sqrt((-3 + x)^2 + (-5 + y
)^2)^0.91))"

julia> eval(Meta.parse(fs))
fz (generic function with 1 method)
```

```
julia> contour(0:0.05:6,0:0.05:6,fz)
```



Trying to minimize $z(x,y)$.

i.e. $f_z(x,y)$ on computer.

Newton's method to solve for $\nabla z = 0$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \nabla z = \begin{bmatrix} \partial z/\partial x \\ \partial z/\partial y \end{bmatrix} = 0$$

Solve $f(\vec{X}) = 0$ where $\vec{X} = \begin{bmatrix} x \\ y \end{bmatrix}$

Big problem $l(t) - f(20) = f'(20)(t - 20)$

$$f_L(X) = f(X_n) + Df(X_n)(X - X_n) = 0$$

Solution X to the above is X_{n+1}.

$$Df\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = D\begin{bmatrix} \partial z/\partial x \\ \partial z/\partial y \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix}$$

Thus

$$f(X_n) + \overbrace{Df(X_n)\, X}^{\text{matrix vect. mult}} = Df(X_n)\, X_n$$

$$Df(X_n)\, X = Df(X_n)\, X_n - f(X_n)$$

So ...

$$X_{n+1} = X_n - (Df(X_n))^{-1} f(X_n)$$

don't find inverse matrix but use the left matrix division "\"

$$X_{n+1} = X_n - Df(X_n) \backslash f(X_n)$$

Input: $x(0)$, $y(0)$, N

Process:

Begin
for $n = 1$ to N do
    Begin
    $q \longleftarrow \partial F/\partial x(x(n-1),\, y(n-1))$
    $r \longleftarrow \partial F/\partial y(x(n-1),\, y(n-1))$
    $s \longleftarrow \partial G/\partial x(x(n-1),\, y(n-1))$
    $t \longleftarrow \partial G/\partial y(x(n-1),\, y(n-1))$
    $u \longleftarrow -F(x(n-1),\, y(n-1))$
    $v \longleftarrow -G(x(n-1),\, y(n-1))$
    $D \longleftarrow qt - rs$
    $x(n) \longleftarrow x(n-1) + (ut - vr)/D$
    $y(n) \longleftarrow y(n-1) + (qv - su)/D$
    End
End

Solving the system

$$X_{n+1} = X_n - Df(X_n) \backslash f(X_n)$$

Output: $x(N)$, $y(N)$

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix} - \left(D \begin{bmatrix} F \\ G \end{bmatrix} \right)^{-1} \begin{pmatrix} F(x(n-1), y(n-1)) \\ G(x(n-1), y(n-1)) \end{pmatrix}$$