Example 4.1. In an unmanaged tract of forest area, hardwood and softwood trees compete for the available land and water. The more desirable hardwood trees grow more slowly, but are more durable and produce more valuable timber. Softwood trees compete with the hardwoods by growing rapidly and consuming the available water and soil nutrients. Hardwoods compete by growing taller than the softwoods can and shading new seedlings. They are also more resistant to disease. Can these two types of trees coexist on one tract of forest land indefinitely, or will one type of tree drive the other to extinction?

Variables:
H = hardwood population (tons/acre)
S = softwood population (tons/acre)
g_H = growth rate for hardwoods (tons/acre/year)
g_S = growth rate for softwoods (tons/acre/year)
c_H = loss due to competition for hardwoods (tons/acre/year)
c_S = loss due to competition for softwoods (tons/acre/year)

Assumptions:
$g_H = r_1 H - a_1 H^2$
$g_S = r_2 S - a_2 S^2$
$c_H = b_1 SH$
$c_S = b_2 SH$
$H \geq 0, S \geq 0$
r_1, r_2, a_1, a_2, b_1, b_2 are positive reals

$a_1 = \frac{r_1}{K_1}$ $\quad a_2 = \frac{r_2}{K_2}$

$b_1 = \frac{r_1 a_1}{K_1}$ $\quad b_2 = \frac{r_2 a_2}{K_2}$

From Quiz 2

$$\boxed{\begin{aligned}\frac{dH}{dt} &= r_1 H\left(1 - \frac{a_1 S + H}{K_1}\right)\\ \frac{dS}{dt} &= r_2 S\left(1 - \frac{S + a_2 H}{K_2}\right)\end{aligned}}$$

Models are the same but sensitivity analysis could be different because the parameters in the model have been rearranged.

$$\frac{dH}{dt} = r_1 H - a_1 H^2 - b_1 SH = r_1 H\left(1 - \frac{a_1}{r_1}H - \frac{b_1}{r_1}S\right)$$

$$\frac{dS}{dt} = r_2 S - a_2 S^2 - bSH$$

to disease. Can these two types of trees coexist on one tract of forest land indefinitely, or will one type of tree drive the other to extinction?

Interpretation of question: Is there an equilibrium state, that is a time-independent solution, such that $H > 0$ and $S > 0$?

Check: Suppose $H(t) = H_0$, $S(t) = S_0$ what values of the constants H_0 and S_0 satisfy the diff. eq.?

$$0 = \frac{dH}{dt} = r_1 H - a_1 H^2 - b_1 S H$$

$$0 = \frac{dS}{dt} = r_2 S - a_2 S^2 - b_2 S H$$

Solve

$$r_1 H - a_1 H^2 - b_1 S H = r_1 H \left(1 - \frac{a_1}{r_1} H - \frac{b_1}{r_1} S\right)$$

$$r_2 S - a_2 S^2 - b_2 S H = r_2 S \left(1 - \frac{a_2}{r_2} S - \frac{b_2}{r_2} H\right)$$

Solution $H_0 = 0$ and $S_0 = 0$ is a solution. But want $H_0 > 0$, $S_0 > 0$

If $S > 0$ then $r_2 \cancel{S} \left(1 - \frac{a_2}{r_2} S - \frac{b_2}{r_2} H\right) = 0$

thus $\frac{a_2}{r_2} S + \frac{b_2}{r_2} H = 1$

If $H > 0$ then

$$r_1 \cancel{H} \left(1 - \frac{a_1}{r_1} H - \frac{b_1}{r_1} S\right) = 0$$

thus $\frac{a_1}{r_1} H + \frac{b_1}{r_1} S = 1$

Two simultaneous equations

$$\frac{a_2}{r_2} S + \frac{b_2}{r_2} H = 1 \qquad X = \begin{bmatrix} H \\ S \end{bmatrix}$$

$$\frac{a_1}{r_1} H + \frac{b_1}{r_1} S = 1$$

Idea is use Cramer's rule:

$$AX = \begin{bmatrix} b_2/r_2 & a_2/r_2 \\ a_1/r_1 & b_1/r_1 \end{bmatrix} \begin{bmatrix} H \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = b$$

So solve $AX = b$. Could just write $X = A \backslash b$.

$$\det A_1 = \det \begin{bmatrix} 1 & a_2/r_2 \\ 1 & b_1/r_1 \end{bmatrix} = \frac{b_1}{r_1} - \frac{a_2}{r_2}$$

$$\det A_2 = \det \begin{bmatrix} b_2/r_2 & 1 \\ a_1/r_1 & 1 \end{bmatrix} = \frac{b_2}{r_2} - \frac{a_1}{r_1}$$

$$\det A = \det \begin{bmatrix} b_2/r_2 & a_2/r_2 \\ a_1/r_1 & b_1/r_1 \end{bmatrix} = \frac{b_2 b_1 - a_1 a_2}{r_1 r_2}$$

Solution is

$$\begin{bmatrix} H_0 \\ S_0 \end{bmatrix} = \begin{bmatrix} \left(\frac{b_1}{r_1} - \frac{a_2}{r_2}\right) \Big/ \left(\frac{b_2 b_1 - a_1 a_2}{r_1 r_2}\right) \\ \left(\frac{b_2}{r_2} - \frac{a_1}{r_1}\right) \Big/ \left(\frac{b_2 b_1 - a_1 a_2}{r_1 r_2}\right) \end{bmatrix}$$

Question what conditions on $a_1, b_1, a_2, b_2, r_1, r_2$ give a positive solution here....

After simplification we find that $H_0 > 0$ and $S_0 > 0$ when

$$\frac{r_2}{a_2} < \frac{r_1}{b_1} \text{ and } \frac{r_1}{a_1} < \frac{r_2}{b_2},$$

see pg. 118 to 119.

will halt the growth of the other type due to competition. The condition for coexistence of both types is that each type reaches the point where it limits its own growth before it reaches the point where it limits the other's growth.

Example 4.2. The blue whale and fin whale are two similar species that inhabit the same areas. Hence, they are thought to compete. The intrinsic growth rate of each species is estimated at 5% per year for the blue whale and 8% per year for the fin whale. The environmental carrying capacity (the maximum number of whales that the environment can support) is estimated at 150,000 blues and 400,000 fins. The extent to which the whales compete is unknown. In the last 100 years intense harvesting has reduced the whale population to around 5,000 blues and 70,000 fins. Will the blue whale become extinct?

$$\frac{dB}{dt} = g_B - c_B = 0.05\,B\left(1 - \frac{B}{150000}\right) - \alpha BF$$

$$\frac{dF}{dt} = g_F - c_F = 0.08\,F\left(1 - \frac{F}{400\,000}\right) - \alpha BF$$

$$X = \begin{bmatrix} B \\ F \end{bmatrix} \qquad G = \begin{bmatrix} 0.05\,B\left(1 - \frac{B}{150000}\right) - \alpha BF \\ 0.08\,F\left(1 - \frac{F}{400\,000}\right) - \alpha BF \end{bmatrix}$$

Thus

$$\frac{dX}{dt} = G(X)$$

Let X_0 be the equilibrium with both types of whales. Then linearize about that equilibrium

$$G(X) \approx G(X_0) + DG(X_0)(X - X_0)$$

and use the linearized equations to check stability... This is called "linear stability"