Note: These notes have been modified from what was presented in class in order to finish the proof that negative eigenvalues of the linearized system imply the distance of the solution to the fixed point decrease over time.

$$\frac{dX}{dt} = G_L(X) \quad \text{where} \quad G_L(X) = G(X_0) + (DG(X_0))(X - X_0)$$

$$\frac{dX}{dt} = G(X_0) + (DG(X_0))(X - X_0)$$

equal

$$\frac{dX_0}{dt} = 0 \quad (X_0 \text{ const}) \qquad \text{and} \qquad G(X_0) = 0$$ by definition of X_0 being the fixed point

$$\frac{d(X - X_0)}{dt} = (DG(X_0))(X - X_0)$$

Write $A = DG(X_0)$ and $y = X - X_0$.

$$\frac{dy}{dt} = Ay$$

note $A \in \mathbb{R}^{2\times 2}$

Now suppose A has two distinct negative eigenvalues λ_1, λ_2 with eigenvectors v_1 and v_2. Thus, the eigenvectors are linearly independent with

$$Av_1 = \lambda_1 v_1 \qquad \text{and} \qquad Av_2 = \lambda v_2.$$

Let $V = \begin{bmatrix} v_1 \vdots v_2 \end{bmatrix}$ be the matrix with columns the eigenvectors.

Then

$$AV = A\begin{bmatrix} v_1 \vdots v_2 \end{bmatrix} = \begin{bmatrix} Av_1 \vdots Av_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 \vdots \lambda_2 v_2 \end{bmatrix} = \begin{bmatrix} v_1 \vdots v_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = V\Lambda$$

where $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is the diagonal matrix of eigenvalues.

So $\quad V^{-1}AV = \Lambda \quad$ and $\quad A = V\Lambda V^{-1}$.

substitute

$$\frac{dy}{dt} = Ay = V\Lambda V^{-1} y$$

Multiply both sides by V^{-1}

$$V^{-1}\frac{dy}{dt} = \cancel{V^{-1}V}\Lambda V^{-1} y$$

to

$$\frac{d\,\overbrace{V^{-1}y}^{z}}{dt} = \Lambda \underbrace{V^{-1}y}_{z} \qquad \text{Now set } \; z = V^{-1}y$$

So

$$\frac{dz}{dt} = \Lambda z$$

and take dot products with z

$$\frac{dz}{dt}\cdot z = \Lambda z \cdot z$$

Since

product rule

$$\frac{d}{dt}\|z\|^2 = \frac{d}{dt}(z\cdot z) = \frac{dz}{dt}\cdot z + z\cdot\frac{dz}{dt} = 2\,\frac{dz}{dt}\cdot z$$

then

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 = \Lambda z \cdot z$$

Now

$$\Lambda z \cdot z = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \lambda_1 z_1^2 + \lambda_2 z_2^2$$

$$\leq \max(\lambda_1, \lambda_2)\left(z_1^2 + z_2^2\right) \leq \max(\lambda_1, \lambda_2)\,\|z\|^2$$

Therefore, setting $\gamma = -2\max(\lambda_1, \lambda_2)$ we have $\gamma > 0$ and

$$\frac{d}{dt}\underbrace{\|z\|^2}_{w} \leq -\gamma \underbrace{\|z\|^2}_{w} \qquad \text{let } w = \|z\|^2$$

Thus $\dfrac{dw}{dt} + \gamma w \leq 0$ mult by $e^{\gamma t}$

$$\frac{dw}{dt}e^{\gamma t} + \gamma e^{\gamma t} w \leq 0$$

$$\frac{d}{dt}\left(w e^{\gamma t}\right) \leq 0$$

$$\int_{t_0}^{t} \frac{d}{dt} w e^{\gamma t}\, dt \leq 0$$

$$w(t)e^{\gamma t} - w(t_0)e^{\gamma t_0} \leq 0$$

so

$$w(t) \leq w(t_0)\, e^{\gamma(t_0 - t)}.$$

$$0 \leq \lim_{t\to\infty} w(t) \leq \lim_{t\to\infty} w(t_0)\, e^{\gamma(t_0-t)} = w(t_0)\cdot 0 = 0$$