Started here

$$\frac{d(x-x_0)}{dt} = \underbrace{(DG(x_0))}_{A}\underbrace{(x-x_0)}_{y}$$

$$\frac{dy}{dt} = Ay$$

use eigenvectors and eigenvalues of A to get

So

$$\frac{dz}{dt} = \Lambda z$$

$z = V^{-1}y$

where V is the matrix of eigenvectors

and $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

is diagonal matrix of eigenvalues

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_1 \end{bmatrix}$$

mult it out

$$\frac{dz_1}{dt} = \lambda_1 z_1$$

$$\frac{dz_2}{dt} = \lambda_2 z_2$$

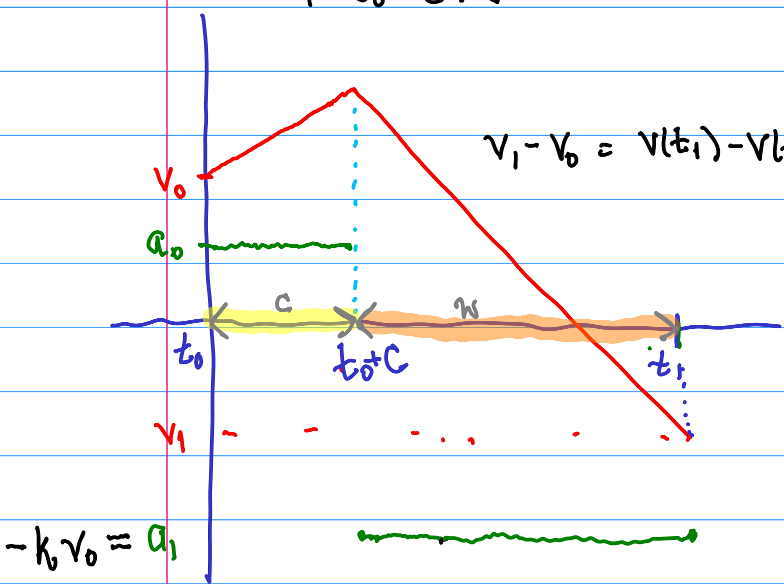
solutions $z_1(t) = c_1 e^{\lambda_1 t}$

$z_2(t) = c_2 e^{\lambda_2 t}$

Example 4.3. Astronauts in training are required to practice a docking maneuver under manual control. As a part of this maneuver, it is required to bring an orbiting spacecraft to rest relative to another orbiting craft. The hand controls provide for variable acceleration and deceleration, and there is a device on board that measures the rate of closing between the two vehicles. The following strategy has been proposed for bringing the craft to rest. First, look at the closing velocity. If it is zero, we are done. Otherwise, remember the closing velocity and look at the acceleration control. Move the acceleration control so that it is opposite to the closing velocity (i.e., if closing velocity is positive, we slow down, and we speed up if it is negative) and proportional in magnitude (i.e., we brake twice as hard if we find ourselves closing twice as fast). After a time, look at the closing velocity again and repeat the procedure. Under what circumstances will this strategy be effective?

$t_1 - t_0 = c + w$

$$\frac{dv}{dt} = a$$



$$v_1 - v_0 = v(t_1) - v(t_0) = \int_{t_0}^{t_1} a(t)\,dt = \int_{t_0}^{t_0+c} a_0\,dt + \int_{t_0+c}^{t_1} a_1\,dt = c a_0 + w a_1$$

$$v_1 = v_0 + c a_0 - w k_0 v_0$$

In general $-k_0 v_n = a_{n+1}$

$$v_{n+1} - v_n = v(t_{n+1}) - v(t_n) = \int_{t_n}^{t_{n+1}} a(t)\,dt = \int_{t_n}^{t_n+c} a_0\,dt + \int_{t_n+c}^{t_{n+1}} a_1\,dt = c a_n + w a_{n+1}$$

$$v_{n+1} = v_n + c a_n - w k v_n = v_n(1 - w k_0) - c k v_{n-1}$$

$$v_{n+1} = v_n(1 - w k_0) - c k v_{n-1}$$

something in the past affects the next velocity...

$$X_n = \begin{bmatrix} v_n \\ v_{n-1} \end{bmatrix}$$

$$X_{n+1} = \begin{bmatrix} v_{n+1} \\ v_n \end{bmatrix} = \begin{bmatrix} v_n(1-wk_0) - ck_0 v_{n-1} \\ v_n \end{bmatrix} = \overbrace{\begin{bmatrix} 1-wk_0 & -ck_0 \\ 1 & 0 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} v_n \\ v_{n-1} \end{bmatrix}}^{X_n}$$

Therefore $X_{n+1} = AX_n$

Find eigenvalues of A. $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 1-wk_0-\lambda & -ck_0 \\ 1 & -\lambda \end{bmatrix} = (1-wk_0-\lambda)(-\lambda) + ck_0 = 0$$

$$\lambda^2 - \lambda(1-wk_0) + ck = 0$$

$\alpha = 1$ $\quad \beta = -(1-wk)$ $\quad \gamma = ck$

$$\lambda = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} = \frac{1-wk_0}{2} \pm \frac{\sqrt{(1-wk)^2 - 4ck_0}}{2} = \lambda_1, \lambda_2$$

So for each eigenvalue there is an eigenvector u_i so

$$Au_1 = \lambda_1 u_1 \quad \text{and} \quad Au_2 = \lambda_2 u_2$$

$$U = [u_1 \vdots u_2]$$

$$AU = A[u_1 \vdots u_2] = [Au_1 \vdots Au_2] = [\lambda_1 u_1 \vdots \lambda_2 u_2] = \overbrace{[u_1 \vdots u_2]}^{U} \overbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}^{D}$$

$$AU = UD \qquad A = UDU^{-1}$$

$$X_{n+1} = AX_n = UDU^{-1}X_n$$

$$\underbrace{U^{-1}X_{n+1}}_{Y_{n+1}} = \cancel{U^{-1}}\cancel{U}DU^{-1}X_n = D\underbrace{U^{-1}X_n}_{Y_n}$$

let $Y_n = U^{-1}X_n$

$$Y_{n+1} = DY_n = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} Y_{n,1} \\ Y_{n,2} \end{bmatrix} = \begin{bmatrix} \lambda_1 Y_{n,1} \\ \lambda_2 Y_{n,2} \end{bmatrix}$$

$$\|Y_{n+1}\|^2 = \|DY_n\|^2 = \underbrace{\lambda_1^2 Y_{n,1}^2 + \lambda_2^2 Y_{n,2}^2}$$

$$a = \max(\lambda_1^2, \lambda_2^2)$$

$$\|Y_{n+1}\|^2 \leq a\left(Y_{n,1}^2 + Y_{n,2}^2\right) = a\|Y_n\|^2$$

If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then $a < 1$,

$$\underbrace{\|Y_1\|^2 \leq a\|Y_0\|^2}$$

$$\|Y_2\|^2 \leq a\|Y_1\|^2 \leq a\left(a\|Y_0\|^2\right) = a^2\|Y_0\|^2$$

$\vdots$ by induction.. (by finding the pattern)

$$0 \leq \|Y_n\|^2 \leq a^n \|Y_0\|^2 \to 0 \quad \text{since } a^n \to 0 \text{ as } n \to \infty$$

since $0 < a < 1$

Then if eigenvalues are less than 1 then

$$\|Y_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

$$\|U^{-1}X_n\| \to 0$$

$$\|X_n\| = \|UU^{-1}X_n\| = \|UY_n\| \leq \|U\|\,\|Y_n\| \to 0$$

So the velocity eventually goes to zero.
and the spaceship stops...