$$L\frac{di}{dt} = -V_c - i^3 - 4i$$

$$C\frac{dV_c}{dt} = i$$

Two equations in two unknowns: $\underbrace{V_c}_{x_2}$ and $\underbrace{i}_{x_1}$.

$$\frac{di}{dt} = \boxed{-\frac{1}{L}\left(V_c + i^3 + 4i\right)} \leftarrow \text{Force}$$

$$\frac{dV_c}{dt} = \frac{i}{C}$$

$$X = \begin{bmatrix} i \\ V_c \end{bmatrix} \qquad F(X) = F\left(\begin{bmatrix} i \\ V_c \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{L}\left(V_c + i^3 + 4i\right) \\ \frac{i}{C} \end{bmatrix}.$$

$$\frac{dX}{dt} = F(X)$$

Linearize this system about fixed points. (way to understand the nonlinear system by solving linear problems).

When is $F(X) = 0$

$$\begin{cases} -\frac{1}{L}\left(V_c + i^3 + 4i\right) = 0 \\ \frac{i}{C} = 0 \end{cases}$$

$-\frac{1}{L}V_c = 0$ means $V_c = 0$

$\leftarrow$ means $i = 0$

only one fixed point when $i = 0$ and $V_c = 0$

linearize

0 because 0 was a fixed point

$$F_L(X) = F(\vec{0}) + DF(\vec{0})(X - \vec{0})$$

$$DF(i, v_c) = D\begin{bmatrix} -\frac{1}{L}\left(v_c + i^3 + 4i\right) \\ \frac{i}{C} \end{bmatrix} = D\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial F_1}{\partial i} & \frac{\partial F_1}{\partial v_c} \\ \frac{\partial F_2}{\partial i} & \frac{\partial F_2}{\partial v_c} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L}(3i^2+4) & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$$

$$DF(0) = \begin{bmatrix} -\frac{4}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} = A$$

up to zero. Detern
e $L = 1$, $C = 1/3$,

linearized equation

$F_L(x) = AX$ so $\frac{dX}{dt} = AX$ where $A = \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix}$

Find eigenvectors and eigenvalues of A.

Characteristic equation $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} -4-\lambda & -1 \\ 3 & -\lambda \end{bmatrix} = \lambda(4+\lambda) + 3 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+3)(\lambda+1) = 0$$ ← generally have to solve a quadratic equation here...

$$\lambda_1 = -3 \qquad \lambda_2 = -1$$

Now for the eigenvectors:

$A u_1 = \lambda_1 u_1$ ← solve for u_1

$(A - \lambda_1 I) u_1 = 0$

$$A - \lambda_1 I = \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$$

Now solve this...

$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} u_1 = 0 \qquad \text{guess} \quad u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{works...}$$

$$A - \lambda_2 I = \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix}$$

Now solve this...

$$\begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} u_2 = 0 \qquad \text{guess} \quad u_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \text{works} \ \boxed{\checkmark}$$

$$Au_2 = \begin{bmatrix} -4 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -4+3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \lambda_2 u_2$$

Use eigenvectors to diagonalize the matrix A.

$$U = \left[\, u_1 \,\vdots\, u_2 \,\right] \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$AU = \left[\, \lambda_1 u_1 \,\vdots\, \lambda_2 u_2 \,\right] = \left[\, u_1 \,\vdots\, u_2 \,\right]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = U\Lambda$$

Thus $A = U\Lambda U^{-1}$ or $\Lambda = U^{-1}AU$

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial

What is $p(A)$? note $A^0 = I$

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$$

$$= a_0 I + a_1 U \Lambda U^{-1} + a_2 (U \Lambda U^{-1})^2 + \cdots + a_n (U \Lambda U^{-1})^n$$

Note

$$A^2 = (U\Lambda U^{-1})^2 = U \Lambda \cancel{U^{-1} U} \Lambda U^{-1} = U \Lambda^2 U^{-1}$$

$$A^0 = U \Lambda^0 U^{-1} = U I U^{-1} = I$$

$$p(U\Lambda U^{-1}) = U(a_0 I + a_1 \Lambda + a_2 \Lambda^2 + \cdots + a_n \Lambda^n) U^{-1} = U p(\Lambda) U^{-1}$$

$$= U\left(a_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_1 \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + a_2 \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} + \cdots + a_n \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}\right) U^{-1}$$

analytic functions... for example e^t, $\sin t$, $\cos t$ ...

By Taylor's theorem many functions can be written as a infinite polynomial (power series)..

$$e^A = e^{U\Lambda U^{-1}} = U e^{\Lambda} U^{-1} = U \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} U^{-1}$$

$$At = U \Lambda U^{-1} t = U(\Lambda t) U^{-1}$$

scalar can move through the matrix mult

eigenvalues of At are t times the eigenvalues of A. The eigenvectors are the same...

$$e^{At} = U \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} U^{-1}$$

$$\frac{dx}{dt} = Ax$$

Solution is $X(t) = e^{\overbrace{At}^{\text{matrix}}} \tilde{c}$

vector of constants

$$X'(t) = Ae^{At}\tilde{c} = AX$$

new constant vector

$c = U^{-1}\tilde{c}$

$$X(t) = e^{At}\tilde{c} = Ue^{\Lambda t}\underbrace{U^{-1}\tilde{c}}_{\text{another constant vector}} = Ue^{\Lambda t}c$$

$$= \begin{bmatrix} u_1 & \vdots & u_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & \vdots & u_2 \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix} = c_1 u_1 e^{\lambda_1 t} + c_2 u_2 e^{\lambda_2 t}$$

These are solutions to the linearized problem...

Next time compare with solutions to the non-linear problem....