$$\frac{dX}{dt} = F(X)$$

$$F(X) = F(x_1, x_2, x_3) = \begin{bmatrix} -\sigma x_1 + \sigma x_2 \\ -x_2 + r x_1 - x_1 x_3 \\ -b x_3 + x_1 x_2 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$F(x,y,z) = \begin{bmatrix} -\sigma x + \sigma y \\ -y + rx - xz \\ -bz + xy \end{bmatrix}$$

Look for fixed points $F(X)=0$. so

$-\sigma x + \sigma y = 0$ $\quad$ $x = y$ at any fixed point

$-y + rx - xz = 0$ $\longrightarrow$ $-x + rx - xz = 0$

$-bz + xy = 0$ $\quad$ $x(r - 1 - z) = 0$

either $x = 0$ or $r - 1 - z = 0$

Case $x = 0$.

then $y = 0$

$-bz + xy = 0$ $\quad$ $-bz = 0$ so $z = 0$

one fixed point is $(x,y,z) = (0,0,0)$

Case $x \neq 0$

then $r - 1 - z = 0$ so $z = r - 1$

$-bz + xy = 0$ $\quad$ $-b(r-1) + xy = 0$

also know $x = y$

$-b(r-1) + x^2 = 0$ $\quad$ $x = \pm\sqrt{b(r-1)}$

note if $r < 1$ then x is imaginary and there are no such fixed points in the phase space. (real values of x,y,z).

If $r>1$ then there are additional fixed points

$$(x,y,z)=\left(\sqrt{b(r-1)},\ \sqrt{b(r-1)},\ r-1\right)$$

and

$$(x,y,z)=\left(-\sqrt{b(r-1)},\ -\sqrt{b(r-1)},\ r-1\right)$$

Check linear. stability of fixed points let X_0 be fixed point then

$$F_L(X)=\underbrace{\cancel{F(X_0)}}_{=0}+DF(X_0)(X-X_0)$$

$$F_L(X)=A(X-X_0)\quad \text{where } A=DF(X_0).$$

Case $X_0=(0,0,0)$

$$F(x,y,z)=\begin{bmatrix}-\sigma x+\sigma y\\ -y+rx-xz\\ -bz+xy\end{bmatrix}$$

$$DF(x,y,z)=\begin{bmatrix}\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z}\\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z}\\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z}\end{bmatrix}=\begin{bmatrix}-\sigma & \sigma & 0\\ r-z & -1 & -x\\ y & x & -b\end{bmatrix}$$

$$A=DF(0,0,0)=\begin{bmatrix}-\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{bmatrix}$$

Check stability by finding the eigenvalues of A and checking whether the real parts of those eigenvalues are pos. or neg.

$$\chi(\lambda)=\det(A-\lambda I)=\det\begin{bmatrix}-\sigma-\lambda & \sigma & 0\\ r & -1-\lambda & 0\\ 0 & 0 & -b-\lambda\end{bmatrix}$$

Expand determinant along the last column

$$\det\begin{bmatrix} -\sigma-\lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{bmatrix} = (-b-\lambda)\det\begin{bmatrix} -\sigma-\lambda & \sigma \\ r & -1-\lambda \end{bmatrix}$$

$$= -(b+\lambda)\left((-\sigma-\lambda)(-1-\lambda) - r\sigma\right)$$

$$= -(b+\lambda)\left((\sigma+\lambda)(1+\lambda) - r\sigma\right)$$

$$= \underbrace{-(b+\lambda)}_{\lambda_3}\left(\lambda^2 + (\sigma+1)\lambda + \sigma - r\sigma\right) = 0$$

one eigenvalue is $\lambda_3 = -b$. ← this eigenvalue has neg. real part assuming $b>0$.

other eigenvalues are roots of $\lambda^2 + (\sigma+1)\lambda - (r-1)\sigma$

$\alpha = 1$ $\quad\beta = \sigma+1$ $\quad\gamma = -(r-1)\sigma$

$$\lambda_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 + 4(r-1)\sigma}}{2}$$

Standard values of $\sigma = 10$ and $b = 8/3$ but leave r.

← plus is more likely positive

$$\lambda_{1,2} = \frac{-(10+1) \pm \sqrt{(10+1)^2 + 4(r-1)10}}{2} = \frac{-11 \pm \sqrt{81+40r}}{2}$$

$$(10+1)^2 + 4(r-1)10 = 11^2 - 40 + 40r = 121 - 40 + 40r = 81 + 40r$$

Stability requires $-11 + \sqrt{81+40r} < 0$

$\sqrt{81+40r} < 11 \qquad 81 + 40r < 121 \qquad 40r < 40 \qquad r < 1.$

If $r > 1$ then the fixed point at $(0,0,0)$ is unstable.
(and there are two other fixed points)

If $r < 1$ then the fixed point at $(0,0,0)$ is stable
(and that's the only fixed point)