Example 7.1. An electronics manufacturer produces a variety of diodes. Quality control engineers attempt to insure that faulty diodes will be detected in the factory before they are shipped. It is estimated that 0.3% of the diodes produced will be faulty. It is possible to test each diode individually. It is also possible to place a number of diodes in series and test the entire group. If this test fails, it means that one or more of the diodes in that group are faulty. The estimated testing cost is 5 cents for a single diode, and $4+n$ cents for a group of $n > 1$ diodes. If a group test fails, then each diode in the group must be retested individually to find the bad one(s). Find the most cost–effective quality control procedure for detecting bad diodes.

$q = 0.3\% = 0.003$

D = random variable representing a diode

$$= \begin{cases} \text{faulty} & \text{with probability } q \\ \text{good} & \text{with probability } p = 1-q \end{cases}$$

B = random variable representing a test batch of n diodes

$$= \begin{cases} \text{all good} & \text{with probability } p^n \\ \text{at least one bad} & \text{with probability } 1-p^n \end{cases}$$

(assume independent failures)

$n=1$ $\quad C = 5$

$n>1$ $\quad C = \begin{cases} 4+n & \text{if } B = \text{all good.} \\ 4+n+5n & \text{if } B = \text{at least one bad.} \end{cases}$

Expected cost per batch is $\mathbb{E}[C]$.

Expected cost per diode is $\mathbb{E}[C/n]$.

$$\mathbb{E}[C] = (4+n)P(B = \text{all good}) + (4+n+5n)P(B = \text{at least one bad})$$

$$= (4+n)p^n + ((4+n)+5n)(1-p^n)$$

$$= 4+n + 5n(1-p^n) = 4+6n-5np^n$$

$$A = \mathbb{E}[C/n] = \frac{\mathbb{E}[C]}{n} = \frac{4+6n-5np^n}{n} = \underbrace{\frac{4}{n}+6-5p^n}_{\text{want to minimize this...}}$$

For simplicity, lets assume n is a continuous variable so we can use calculus to find the minimum

$$\frac{d}{dn}\left(\frac{4}{n}+6-5p^n\right) = \underbrace{-\frac{4}{n^2}-5(\ln p)p^n}_{} = 0$$

solve for n using Newton's method

$$\frac{d}{dn}p^n = \frac{d}{dn}e^{n\ln p} = (\ln p)e^{n\ln p} = (\ln p)p^n$$

$$\frac{4}{n^2} = 5\left(\ln\frac{1}{p}\right)p^n$$

$$\ln 4 - 2\ln n = \ln\left[5\left(\ln\frac{1}{p}\right)\right] + n\ln p$$

$$\ln 4 - \ln\left[5\left(\ln\frac{1}{p}\right)\right] = \underbrace{2\ln n - n\ln p}_{}$$

again need Newton's method to solve for n.

alternatively just plug in values of n and check the costs...

To minimize the cost we use expected cost so the per-diode cost function was no longer a random variable...

Statistics $\Rightarrow$ estimator like expected value $\Rightarrow$ calculus

or something like making a table...

Another approach is to simulate the random variables directly on the computer and look to see what the simulation says...

If $q = 0.003$ then optimal batch size is $n_{opt} = 17$

and the lowest per-diode cost is $A_{opt} =$ 1.484264961220581

Let's compute the relative sensitivity...

$$S(A,q) = \frac{q}{A}\frac{dA}{dq}\Bigg|_{n=n_{opt},\, A=A_{opt}.}$$

$$\frac{dA}{dq} = \frac{d}{dq}\left(\frac{4}{n} + 6 - 5p^n\right) = \frac{d}{dq}\left(\frac{4}{n} + 6 - 5(1-q)^n\right) = 5(1-q)^{n-1}$$

$$S(A,q) = \frac{q}{A}\frac{dA}{dq}\Bigg|_{n=n_{opt},\, A=A_{opt}.} = \frac{5(1-q)^{n-1}q}{\frac{4}{n} + 6 - 5(1-q)^n}\Bigg|_{\substack{q=0.003 \\ n=n_{opt},\, A=A_{opt}}}$$