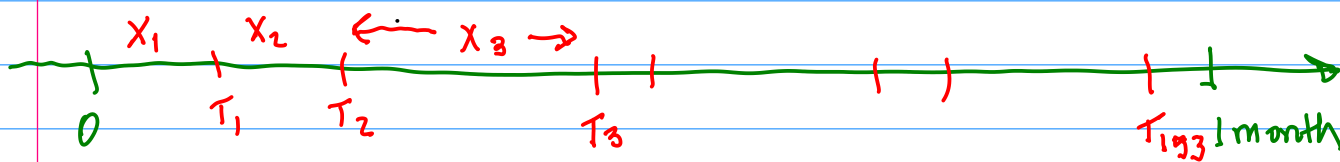
Example 7.4. An emergency 911 service in a local community received an average of 171 calls per month for house fires over the past year. On the basis of this data, the rate of house fire emergencies was estimated at 171 per month. The next month only 153 calls were received. Does this indicate an actual reduction in the rate of house fires, or is it simply a random fluctuation?

rate of fires $\lambda = 171$ per month.



time between fires

X_i's are distributed according to a exponential distribution $\lambda = 171$ (and independence).

$T_n = X_1 + X_2 + \cdots + X_n$

law of large numbers

X is exponential with λ

$$\frac{T_n}{n} \longrightarrow E[X] = \frac{1}{\lambda}$$

Note X has density

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$E[X] = \int_0^\infty t \lambda e^{-\lambda t}\, dt = t\left(-e^{-\lambda t}\right)\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\, dt$$

$u = t \qquad dv = \lambda e^{-\lambda t} dt$

$du = dt \qquad v = -e^{-\lambda t}$

$$= \lim_{t \to \infty} t\left(-e^{-\lambda t}\right) - 0\left(-e^{-\lambda 0}\right) + -\frac{1}{\lambda} e^{-\lambda t}\Big|_0^\infty = \frac{1}{\lambda}$$

Central Limit Theorem:

$$\mathbb{P}\left[\frac{(T_n - n\mu)}{\sigma\cdot\sqrt{n}} \leq \varepsilon\right] \rightarrow \Phi(\varepsilon) \quad \text{as} \quad n \to \infty$$

where $T_n = X_1 + X_2 + \cdots + X_n$ and X_i's are independent identically distributed random variables and

$$\Phi(\varepsilon) = \int_{-\infty}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

cumulative distribution function of a standard Gaussian (or normal) random variable

$$\mu = \mathbb{E}[X]$$

$$\sigma^2 = \mathbb{V}[X] = \mathbb{E}[(X-\mu)^2]$$

Note...

$\mathbb{E}[\mu^2] = \mu^2$

const.

For exponentially distributed random variables

$$\mu = \frac{1}{\lambda}$$

$$\sigma^2 = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu\underbrace{\mathbb{E}[X]}_{\mu} + \mu^2$$

plug in for μ.

$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 = \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X^2] = \int_0^\infty t^2 \lambda e^{-\lambda t}\, dt$$

$$\mathbb{E}[X^k] = \int_0^\infty t^k \lambda e^{-\lambda t}\, dt$$

$u = t^k \qquad dv = \lambda e^{-\lambda t}\, dt$

$du = k t^{k-1}\, dt \qquad v = -e^{-\lambda t}$

assume $k > 0$, $\lambda > 0$

$$= -\cancel{t^k}\,\cancel{e^{-\lambda t}}\Big|_0^\infty + \int_0^\infty k t^{k-1} e^{-\lambda t}\, dt$$

$$= \frac{k}{\lambda} \int_0^\infty t^{k-1} \lambda e^{-\lambda t}\, dt = \frac{k}{\lambda} \mathbb{E}\left[X^{k-1}\right]$$

Therefore

$$\mathbb{E}\left[X^k\right] = \frac{k}{\lambda} \mathbb{E}\left[X^{k-1}\right] \qquad \mu = \mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathbb{E}\left[X^2\right] = \frac{2}{\lambda} \mathbb{E}[X] = \frac{2}{\lambda^2}$$

remark ...

$$\mathbb{E}\left[X^k\right] = \frac{k!}{\lambda^k}$$

$$\sigma^2 = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

For X_i's being exponentially distributed with rate λ then

$$\mathbb{P}\left[\frac{\left(T_n - n\frac{1}{\lambda}\right)}{1/\lambda\sqrt{n}} \leq \varepsilon\right] \approx \Phi(\varepsilon)$$

for n large by the central limit theorem...

$$\mathbb{P}\left[\frac{\left|T_n - n\frac{1}{\lambda}\right|}{1/\lambda\sqrt{n}} \leq \varepsilon\right] = \mathbb{P}\left[-\varepsilon \leq \frac{\left(T_n - n\frac{1}{\lambda}\right)}{1/\lambda\sqrt{n}} \leq \varepsilon\right]$$

$$= \mathbb{P}\left[\frac{\left(T_n - n\frac{1}{\lambda}\right)}{1/\lambda\sqrt{n}} \leq \varepsilon\right] - \mathbb{P}\left[\frac{\left(T_n - n\frac{1}{\lambda}\right)}{1/\lambda\sqrt{n}} < -\varepsilon\right]$$

$$\approx \Phi(\varepsilon) - \Phi(-\varepsilon)$$

how big this is?

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\,dx \qquad \boxed{\mathrm{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2}\,dx}$$

Standard functions built in to many programming languages

$$\Phi(\varepsilon) = \int_{-\infty}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt = -\int_{\infty}^{-\varepsilon/\sqrt{2}} \frac{1}{\sqrt{2}\sqrt{\pi}} e^{-u^2} \sqrt{2}\,du$$

$u = -t/\sqrt{2}$ $\qquad du = -dt/\sqrt{2}$

$u^2 = t^2/2$

$$= \int_{-\varepsilon/\sqrt{2}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2}\,du = \frac{1}{2}\mathrm{erfc}(-\varepsilon/\sqrt{2})$$

$$\mathbb{P}\left[\frac{|T_n - n\frac{1}{\lambda}|}{1/\lambda\sqrt{n}} \leq \varepsilon\right] \approx \Phi(\varepsilon) - \Phi(-\varepsilon)$$

$$= \frac{1}{2}\mathrm{erfc}(-\varepsilon/\sqrt{2}) - \frac{1}{2}\mathrm{erfc}(-\varepsilon/\sqrt{2})$$

Try some values for ε.

```
julia> Phi(epsilon)=1/2*erfc(-epsilon/sqrt(2))
Phi (generic function with 1 method)

julia> using SpecialFunctions

julia> Phi(1)-Phi(-1)
0.6826894921370859
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$\varepsilon = 1$ $\qquad \Phi(1) - \Phi(-1) = 0.6826894921370859 \approx 68\%$

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julia> Phi(2)-Phi(-2)
0.9544997361036416
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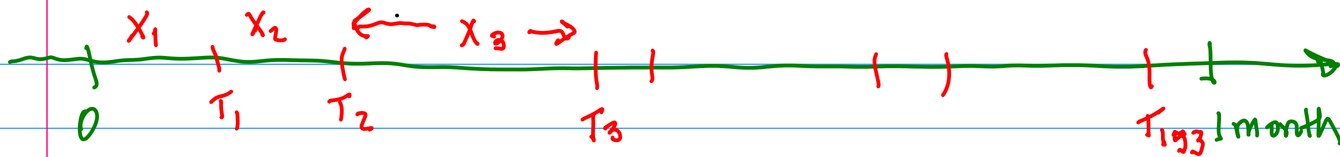
$\varepsilon = 2$ $\qquad \Phi(2) - \Phi(-2) = 0.9544997361036416 \approx 95\%$

$\lambda = 171$ and $\mu = \frac{1}{\lambda} = \frac{1}{171}$ $\sigma = \frac{1}{\lambda} = \frac{1}{171}$



Next time plug in and... $n = 153$

$$\frac{\left|T_n - n\frac{1}{\lambda}\right|}{1/\lambda\sqrt{n}} =$$ and compare...