$$P = \begin{bmatrix} .367 & 0 & .632 \\ .367 & .367 & .264 \\ .184 & .363 & .448 \end{bmatrix}$$

$S_0 = 3$ start with three aquariums

State $\pi_0 = (0, 0, 1)$

probability in state 1

prob. in state 2

prob. in state 3,

```
julia> ps=[0 0 1]
1×3 Matrix{Int64}:
 0  0  1

julia> ps*P
1×3 Matrix{Float64}:
 0.18394  0.367879  0.448181
```

Next week $\pi_1 = (0.184, 0.367, 0.448)$

$P(S_1 = 1 \mid S_0 = 3) \cdot 1 = 0.184$

$P(S_1 = 2 \mid S_0 = 3) \cdot 1 = 0.367$

$P(S_1 = 3 \mid S_0 = 3) \cdot 1 = 0.448$

2nd week $\pi_2 = \big(P(S_2 = 1),\ P(S_2 = 2),\ P(S_2 = 3) \big)$ (row vector)

$$P(S_2=1) = \underbrace{P(S_2=1 \mid S_1=1)}_{P_{12}} \cdot \underbrace{P(S_1=1)}_{\pi_1} + \underbrace{P(S_2=1 \mid S_1=2)}_{P_{21}} \cdot \underbrace{P(S_1=2)}_{\pi_2} + \underbrace{P(S_2=1 \mid S_1=3)}_{P_{31}} \cdot \underbrace{P(S_1=3)}_{\pi_3}$$

$$P(S_2=2) = \underbrace{P(S_2=2 \mid S_1=1)}_{P_{12}} \cdot \underbrace{P(S_1=1)}_{\pi_1} + \underbrace{P(S_2=2 \mid S_1=2)}_{P_{22}} \cdot \underbrace{P(S_1=2)}_{\pi_2} + \underbrace{P(S_2=2 \mid S_1=3)}_{P_{32}} \cdot \underbrace{P(S_1=3)}_{\pi_3}$$

$$P(S_2=3) = \underbrace{P(S_2=3 \mid S_1=1)}_{P_{13}} \cdot \underbrace{P(S_1=1)}_{\pi_1} + \underbrace{P(S_2=3 \mid S_1=2)}_{P_{23}} \cdot \underbrace{P(S_1=2)}_{\pi_2} + \underbrace{P(S_2=3 \mid S_1=3)}_{P_{33}} \cdot \underbrace{P(S_1=3)}_{\pi_3}$$

$$P_{12}\cdot\pi_1 + P_{21}\cdot\pi_2 + P_{31}\cdot\pi_3$$
$$P_{12}\cdot\pi_1 + P_{22}\cdot\pi_2 + P_{32}\cdot\pi_3$$
$$P_{13}\cdot\pi_1 + P_{23}\cdot\pi_2 + P_{33}\cdot\pi_3$$

first index of P_{ij} agrees with the vector index.

$$\pi_1\cdot P_{12} + \pi_2\cdot P_{21} + \pi_3\cdot P_{31}$$
$$\pi_1\cdot P_{12} + \pi_2\cdot P_{22} + \pi_3\cdot P_{32}$$
$$\pi_1\cdot P_{13} + \pi_2\cdot P_{23} + \pi_3\cdot P_{33}$$

now nearest indices agree.

Viewing π as a row vector $\pi\in\mathbb{R}^{1\times 3}$ then

$$\pi_{n+1} = \pi_n P$$

```
julia> P=[exp(-1) 0 1-exp(-1);
          exp(-1) exp(-1) 1-2*exp(-1);
          1/2*exp(-1) exp(-1) 1-(1+1/2)*exp(-1)]
3×3 Matrix{Float64}:
 0.367879  0.0       0.632121
 0.367879  0.367879  0.264241
 0.18394   0.367879  0.448181
```

```
julia> pn=ps
       for n=1:10
           pn=pn*P
           println(pn)
       end
[0.18393972058572117 0.36787944117144233 0.4481808382428365]
[0.2854411830131807 0.300211799553136 0.4143470174336833]
[0.29166456655916373 0.262871498277238 0.4546393516359983]
[0.2859409294064343 0.2605820434161462 0.4534770271774195]
[0.28446700350038434 0.26268765185336046 0.45284534464625525]
[0.2845831950086656 0.2632298788920062 0.4521869260993282]
[0.2847043043322158 0.26318713442487085 0.45210856124291343]
[0.28471871874200844 0.26314258079460257 0.45213870046338905]
[0.2847131749422155 0.263137278029583 0.4521495470282013]
[0.28471117982811456 0.263139317479553 0.4521495026923324]
```

Probabilities of having 1, 2 or 3 aquariums in inventory at the beginig of the 10th week.

Note that π_n appears to converge as $n \to \infty$.

```
julia> pn=ps
       for n=1:100
           pn=pn*P
           #println(pn)
       end

julia> pn
1×3 Matrix{Float64}:
 0.284711  0.26314  0.452149

julia> println(pn)
[0.28471134182344127 0.2631399918462632 0.4521486663302953]
```

```
julia> print(pn*P)
[0.28471134182344127 0.2631399918462632 0.4521486663302953]
```

exactly the same...

$\pi = \pi P$ ← find the fixed point directly..

also need $\pi_1 + \pi_2 + \pi_3 = 1$ and $\pi_1, \pi_2, \pi_3 \geq 0$

Convert to eigenvalue/eigenvector problem

$$\pi^T = (\pi P)^T = P^T \pi^T$$

$$P^T \pi^T = \underbrace{\lambda}_{1} \pi^T$$

so π^T is an eigvector of P^T with eigenvalue $\lambda = 1$.

```
julia> eigvals(P')
3-element Vector{ComplexF64}:
 0.09196986029286056 - 0.2433293784482631im
 0.09196986029286056 + 0.2433293784482631im
   1.0000000000000001 + 0.0im
```

← $\lambda = 1$

```
julia> eigvecs(P')
3×3 Matrix{ComplexF64}:
 -0.158114-0.41833im  -0.158114+0.41833im  -0.478023+0.0im
 -0.474342+0.41833im  -0.474342-0.41833im  -0.441805+0.0im
  0.632456-0.0im       0.632456+0.0im      -0.759146+0.0im
```

eigenvector corresponding to $\lambda = 1$

```
julia> println(real(eigvecs(P')[:,3]))
[-0.4780228927523713, -0.4418051606078647, -0.7591457791919242]
```

```
julia> print(pn*P)
[0.2847113418234412 7 0.2631399918462632 0.4521486663302953]
julia>
```

expected this...

also need $\pi_1 + \pi_2 + \pi_3 = 1$ and $\pi_1, \pi_2, \pi_3 \geq 0$

```
julia> v=-real(eigvecs(P')[:,3])
3-element Vector{Float64}:
 0.4780228927523713
 0.4418051606078647
 0.7591457791919242

julia> sum(v)
1.6789738325521602

julia> v/sum(v)
3-element Vector{Float64}:
 0.2847113418234412 7
 0.2631399918462632
 0.45214866633029555
```

fix so $\pi_1 \geq 0$ $\pi_2 \geq 0$ $\pi_3 \geq 0$

rescale so $\pi_1 + \pi_2 + \pi_3 = 1$ ✓

rounding difference...

Question: How many customers on average enter the store but can not purchase an aquarium because they are out of stock?